Hydrodynamics as scaling limit of kinetic systems and stochastic particle systems.

by

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ABSTRACT. We discuss and review some main results on the derivation of Euler and Navier-Stokes equations from microscopic dynamics: we restrict our attention to two kinds of microscopic dynamics: the Boltzmann equation and some stochastic particle systems. We introduce the concept of hydrodynamic limit and show how to prove in such a limit the convergence of kinetic systems to macroscopic equations in various situations such as in presence of an external force or in a bounded domain both in time-dependent or stationary case. We introduce systems of particles moving according to some natural stochastic dynamics, in the real space or on the lattice, and discuss the relevance of a new probabilistic method, called non-gradient method, for proving the hydrodynamic limit on the diffusive scale and for getting the transport coefficients in terms of the Green-Kubo formulas.

Contents:

1) Introduction.
2) Boltzmann equation and hydrodynamic limits.
3) Hamiltonian systems: heuristics.
4) Lattice Gases.
5) Navier-Stokes equation.
6) Non gradient and entropy methods.

1. INTRODUCTION.

The time evolution of complex systems, and in particular fluids, can be described at least on two different levels: the macroscopic level, based on phenomenological partial differential equations relating a relatively small number of variables (mass density, specific momentum and energy of a fluid) and involving some empirically determined parameters (specific heats, viscosity coefficients, heat conductivity etc. for fluids); the microscopic level, based on some fundamental law (Newton or Schröedinger equations) and a suitable molecular interaction, involving a huge number of variables, namely the positions and velocities of all the molecules, but no empirical parameters.

The macroscopic description, called *hydrodynamical description* when referred to fluids, has clear advantages, but the choice of the right phenomenological equations and of the empirical parameters is a problem that has to be solved in every specific situation and often in an unsatisfactory way from the conceptual point of view.

On the other hand, the microscopic description, although conceptually very clean and free of empirical parameters, is essentially useless from the practical point of view because of the enormous amount of variables involved.

It is therefore very natural to ask the question whether it is possible to obtain the phenomenological equations to be used in the macroscopic description by means of some systematic procedure based on the microscopic description and to compute the empirical parameters exclusively in terms of the mass of the molecules and the molecular interaction.

It is clear that the task of constructing such a procedure is a very difficult one for several reasons that we will examine in this paper. For the moment we concentrate our attention on the following remark: the macroscopic equations are of deterministic nature, namely given initial value to the relevant variables, the evolution of the system is completely determined. On the other hand, the microscopic evolution depends on the initial positions and velocities of the molecules. Since it is impossible to prepare carefully the initial configuration of a large number of particles, one could expect that the evolution of the microscopic system, due to the instabilities present in the equations of motion even for systems of few particles, appear to be essentially erratic and hardly connected to any macroscopically deterministic behavior.

The connection between the macroscopic and microscopic description is instead quite strong and the link is provided exactly by the large number of particles, but for a rather subtle reason: the arising of a *law of large numbers*, which forces variables depending on a large number of random components, to be well approximated by deterministic (almost sure) variables. This means that the relation between the microscopic and macroscopic descriptions can be really obtained only asymptotically, as the number of molecules goes to infinity.

A convenient way to make more precise previous arguments is to introduce the notion of scaling limit. This is based on the fact that there are two natural length scales in the system: a microscopic length, say the range of the interaction, and a macroscopic scale, say the size of the container. We call ε the ratio between these two scales. If we adopt the microscopic length as unit of measure of the lengths, then the size of a macroscopic region Δ is of order ε^{-1}. If we assume that the mass density of the fluid is finite, then the number of particles in the region Δ is of order ε^{-d}, d being the number of space dimensions (3 in the physical situations). Therefore, asymptotics for large numbers of particles is equivalent to the asymptotics for $\varepsilon \to 0$, which is referred to as *scaling limit*. We refer to such a situation as *scale separation*.

The law of large numbers or equivalently the scale separation provides a possible mechanism to obtain macroscopic equations, but does not single out the relevant variables. In fact they have to be selected by the fundamental equations: among all the variables associated to a microscopic system some are special, the locally conserved quantities. If one considers the interaction between two particles thought as alone, the mass, momentum and energy are conserved. For a fluid, the mass, momentum and energy of a small macroscopic part of the fluid, are the *locally conserved quantities* because, although they are not exactly conserved, we expect that their variations in time are much smaller that those of all the other variables, which fluctuate much more, due to the instabilities of the dynamics and do not have any meaningful macroscopic behavior. Therefore the hydrodynamical equations are combinations of two deep features of the system: scale separation and local conservation laws.

Using above ingredients it is possible to provide convincing formal derivations of hydrodynamical equations in several situations and we will describe a few of them. To get rigorous proofs of such derivations is a completely different matter. At the moment, with some remarkable but peculiar exception, the derivation of hydrodynamical equations for deterministic microscopic models is a completely open problem, probably outside the range of the present mathematical techniques. For this reason it is desirable to have simpler models where one can test the effectiveness of above ideas. In the last few years the efforts have been concentrated essentially in two directions: rigorous derivations of the hydrodynamical equations from the Boltzmann equation and from stochastic particle systems.

The Boltzmann equation arises when a rarefaction assumption is made on the system, namely that the mass density ρ is small. In this case a new length scale is present, the typical distance a particle can cross without undergoing collisions, called *mean free path*. Clearly, the smaller the density, the larger the mean free path in microscopic unities. For a rarified gas in suitable initial conditions Boltzmann [4] in 1872 was able to derive an evolution equation for the probability density of particles in the phase space, the Boltzmann equation and Lanford [33] proved rigorously the validity of such a derivation in 1975. Since the mean free path is typically much smaller that the macroscopic length, it makes sense to try to derive the hydrodynamical equations starting from the Boltzmann equation rather than from the particle system itself. This has its intrinsic interest because in some situations it is correct to describe the system using the Boltzmann equation. Furthermore several of the difficulties one has to face for the derivation of hydrodynamical equations from particle systems are captured by the Boltzmann equation and one can hope to separate the difficulties by looking at the problem at the Boltzmann level.

Stochastic particle systems are often useful to study complicated situations where it is not easy to use models based only the fundamental laws of Physics. Hence it is interesting from the practical point of view to be able to describe the collective behavior of such systems on the macroscopic scale. On the other hand the effect of the instabilities present in the deterministic dynamics could be simulated by some suitable noise and one could take advantage, from the technical point of view, of such stochasticity. For this reason in the recent years a big effort has been concentrated on the analysis of these models.

Ideas and techniques in the first approach have been successfully exported in the second one and interesting analogies have been used. The aim of this paper is to illustrate the recent results obtained and the mutual influence among the two approaches.

In the derivation of hydrodynamical equations a major role is played by the notion of *local equilibrium* state.

A (global) equilibrium state, both for the Boltzmann equation and for the particle systems, is a measure on the phase space of the system, solving the dynamical problem, which is stationary in time and translationally invariant. The classification of such measures is rather simple for the Boltzmann equation where equilibria turn out to be a family of functions, called Maxwellians, given explicitly by

$$M(\rho, u, T; v) = \frac{\rho}{\sqrt{(2\pi T)^d}} \exp[-\frac{(v-u)^2}{2T}].$$

Here $\rho > 0$, $T > 0$ and $u \in \mathbb{R}^d$, denoted globally by ζ are $d+2$ parameters which are interpreted as *mass density*, *velocity field* and *temperature* of the system at the equilibrium. They do not depend on the space coordinate because of the translation invariance. For particle systems the structure of the equilibrium states is more complex because of the interaction, but still the equilibria, called *Gibbs states or measures* can be parametrized by $d{+}2$ parameters with the same interpretation as before. An important difference is that for particle systems in some regions of the parameters space the correspondence between the values of the parameters and the Gibbs states may not be one to one because of the presence of *phase transitions*. The discussion of such phenomena is beyond the purposes of this paper, so we will always assume to be in a region of the parameters where phase transitions are absent.

A local equilibrium corresponds to assume that each given macroscopic region of the system is in equilibrium, but different regions may be in different equilibrium states, corresponding to different values of the parameters. This amounts to assume that the parameters, instead of being independent of the microscopic coordinate $r \in \mathbb{R}^d$, depend on r in such a way to have sensitive variations only on distances of macroscopic size. This means that if $x \to \zeta(x) \in \mathbb{R}^{d+2}$ is a smooth function, then $\zeta^\varepsilon : r \to \zeta(\varepsilon r)$ describes the space dependence of the parameters. For the Boltzmann equation the local equilibria are obtained just by replacing the constant parameters with above varying parameters. For particle systems the structure of the local equilibria will be discussed extensively in Section 3.

In all the cases local equilibrium is never an exact solution of the evolution problem, but depending on the circumstances it may be a good approximation in the limit $\varepsilon \to 0$. This introduces another important ingredient that we have up to now disregarded: the time scale. Indeed, the behavior of the system is very different on different time scales and we are going to sketch the essential points. To this end we will assume that the initial distribution of the system is given by a local equilibrium. This is not essential, because any state varying slowly with r would become close to equilibrium after a very short time (on the macroscopic scale), but we assume it to simplify the discussion.

Let us fix some positive time τ. After this time a given particle, moving with finite velocity of size v spans a space region of size $v\tau$. Since the parameters of the equilibrium are essentially constant on distances of order ε^{-1}, if $v\tau \ll \varepsilon^{-1}$ it has no chance to interact with particles distributed according to a different equilibrium and therefore, at time τ the initial local equilibrium is presumably still a good approximation of the solution. The approximation becomes worse with the time increasing and becomes definitely not good when $v\tau \approx \varepsilon^{-1}$. At such a time one could expect that the local equilibrium structure is completely destroyed and the distribution of the system becomes complicate. A more optimistic approach would suggest that equilibrium is so stable that it persists in each macroscopic region but with suitably modified values of the parameters. Suppose that $\tau = \varepsilon^{-1}t$, so that the condition $v\tau \approx \varepsilon^{-1}$ is fulfilled. The local equilibrium would then be given by parameters $\zeta^{\varepsilon} = \zeta(\varepsilon r, t)$ with $(x, t) \to \zeta(x, t)$ some suitable smooth function. The assumption that the state is "well approximated" at time $t > 0$ by a suitable local equilibrium state is the *local equilibrium assumption*. The time scale $\tau = \varepsilon^{-1}t$ is called *hyperbolic* or *Euler* time scale, when space is simultaneously scaled as $r = \varepsilon^{-1}x$. Once one assumes local equilibrium, the next question is how one should choose the functions $(x, t) \to \zeta(x, t)$. The criterion to fix them is to get such an approximation of the true state that the error goes to 0 as $\varepsilon \to 0$. It turns out that, in order to fulfill such a condition, the parameters ζ have to solve the *Euler equations* for a perfect fluid. This is quite easy to realize for the Boltzmann equation, as discussed in Section 2, and somehow more involved for particle systems. This will be discussed in Section 3, where the appropriate references will also be presented.

The local equilibrium assumption, although reasonable, requires to be justified. The first attempt in this direction is due to Hilbert in 1912 [25], who solved the Boltzmann equation in terms of a power series in ε, the *Hilbert expansion*, whose lowest order term is exactly the local equilibrium satisfying the Euler equation for an ideal gas. No information on the convergence of the series is available and it was only in 1980 that Caflisch [6] was able to prove that a truncated Hilbert expansion approximates the solution to the Boltzmann equation (at least for all the times when the Euler equations have smooth solutions) thus providing a rigorous derivation of the Euler equations from the Boltzmann equation.

For a system of classical particles evolving according to the Newton equations with some nice interaction, nobody has been able up to now to prove the correctness of the local equilibrium assumption. In fact this involves the understanding of deep mechanisms related to the instabilities of the microscopic dynamics that are hard to achieve even for much simpler systems. The study of such problems is the object of the *Ergodic theory* that is not among the issues we are going to deal with in this paper. In this situation the introduction of some artificial noise turns out to be very helpful. The justification of such a noise, as mentioned before, is due to the fact that one expects erratic behaviors of the microscopic dynamics whose effective result may be simulated by a suitable noise to be added to the system. Of course, there is no rigorous justification of this starting from the fundamental laws, the technical convenience being the main justification. Thanks to the introduction of random noises in an otherwise deterministic Hamiltonian particle system, in [36] it has been possible to achieve a full justification of the local equilibrium assumption thus getting a rigorous proof of the Euler equations for a fluid from a stochastic particle system. The method employed in [36] is the Yau's entropy method introduced in [44]. It is a very convenient tool to prove the closeness of the distribution of the system to the appropriate local equilibrium. In Section 6 we shall present an extensive discussion of the method, which is helpful also in dealing with the derivation of the Navier-Stokes equations. This is our next issue.

Euler equations, as well known, describe non viscous fluids and have intrinsic inconsistencies (D'Alambert paradox just to quote one of them) that are fixed by the inclusion of viscosity in the model so getting the Navier-Stokes-Fourier equations. In the scaling limit approach viscous and heat conducting effects arise on a longer time scale, compared with the Euler scale: they depend on second order space derivatives which are of order ε^2 while the first order space derivatives are of order ε, due to the space scaling. Hence these effects are appreciable when $\tau = \varepsilon^{-2}t$. The space-time scaling $\tau = \varepsilon^{-2}t$, $r = \varepsilon^{-1}x$ is called *parabolic* or *diffusive* or *Navier-Stokes* scaling, depending on the context where it is used.

In the diffusive scaling new hard problems arise. In fact, on so long times the errors tend to accumulate and the small error done in the Euler scale by replacing the true solution with the local equilibrium may become finite. This means that the local equilibrium assumption is not sufficient to get the hydrodynamical equations on the diffusive scale and one has to get a more accurate approximation. The simpler case where one can realize how to obtain viscous effect is the Boltzmann equation: the first correction to the local equilibrium in this case can be easily computed via the Hilbert expansion or a suitable resummation of it, known as Chapman-Enskog expansion. Actually the main success of the kinetic theory has been getting the expressions of the transport coefficients in terms of the molecular interaction and its correct dependence on the density (see [7] and references quoted therein).

A difficulty that one encounters in trying to derive the Navier-Stokes equations is that it involves simultaneously transport effects described by first order differential operators and diffusion phenomena described by second order differential operators. As noted before, the first ones scale naturally according to the hyperbolic scaling while the second ones scale according to the parabolic scaling. As a consequence, it is impossible in general to have them to coexist in a scaling limit procedure as the one outlined before.

Indeed, there is no general derivation of the Navier-Stokes equation from a microscopic model as scaling limit, even at heuristic level. The previously quoted results merely deal with the computation of the first order correction to the local equilibrium, without attempting to show the convergence of the scaling limit: on the Euler time scale the limit are the Euler equations, on the Navier-Stokes scale, the limit probably does not exist in general.

The only situations that can be handled as scaling limit are those where the transport and diffusion terms in the equations are of the same order in ε. A few remarkable examples of such a situation do exist and we will detail about them in Section 2. The simplest one corresponds to the *incompressible Navier-Stokes equation* (INS). The incompressible regime in the motion of a fluid is determined when the *Mach number* i.e. the ratio between the typical particle speed and some characteristic velocity, e.g. the sound velocity, is small (*low Mach numbers limit*). When performing a scaling limit, one can also assume special initial data where the Mach number is of order ε. This assumption extends to positive times thus providing a transport operator of the same size of the diffusion operator. We have now a scaling invariant situation.

In this case a modified Hilbert expansion can be set up and the proof of the convergence of the rescaled Boltzmann equation to the INS can be achieved. In Section 2 we present a more detailed discussion of this result and the relevant references on the subject.

The approach used in the kinetic framework can be successfully extended to Hamiltonian particle systems, at least at formal level, and it is possible to construct a kind of modified Hilbert expansion of the solution to the Liouville equation, which allows to compute formally the first correction to the local equilibrium and thus obtain the expressions of transport coefficients in terms of the microscopic interaction. These expressions are usually obtained by a different approach and are known as the Green-Kubo formulas. This is discussed in Section 3. An important assumption for obtaining, at least formally, this result is that the null space of the Liouville operator is spanned only by the mass, momentum and energy, as for the linear Boltzmann operator, an assumption very hard to check.

To give a mathematical sense to the computation of the correction to the local equilibrium is much harder than in the kinetic case, where it is just matter of Fredholm alternative. A crucial condition to get finite transport coefficients is in fact a strong decay property of the time autocorrelations of mass, momentum and energy currents. For deterministic particle systems there is no clue how to do that and even adding suitable stochasticity, as in [36], does not help to solve the problem, at least for the moment. The only situations where it is known how to construct such a correction are confined to models where the space structure is discrete. For this reason we are forced to abandon Hamiltonian particle systems and switch to stochastic lattice gases as we do in Section 4. While the addition of stochasticity was motivated by some reasonable argument, the lattice assumption is only matter of convenience: we cannot do nothing in the case of real space, but we hope to get some insight on the problem by looking at models where the mathematical treatment is doable, although not easy.

A very simple and interesting example of stochastic lattice gas is provided by the simple exclusion process where each particle can jump, according to a Poisson time, to a neighboring site independently of the other particles, provided that the target site is not already occupied. If the probability of jumping in opposite directions is the same, than this is the symmetric simple exclusion process that converges, in the diffusive limit, to the heat equation.

When the jumps in opposite directions are not equally likely, the model becomes much more complicated. First of all the hyperbolic scaling is already relevant because transport phenomena arise. The limiting behavior in the hyperbolic scaling of this process is well understood and is given by the Burgers equation even when it develops singularities [38], a remarkable and rare situation. The techniques used to get this result are quite different from those we are discussing here, so we refer to the original paper for this result.

In the diffusive scaling the same problem mentioned before for the Navier-Stokes equations is present for the simple exclusion process and the limiting behavior cannot be obtained in general, but only in the analog of the low Mach numbers situations: the initial density profile of the system is assumed to deviate from the constant one only for terms of order ε.

Under this scaling a viscous Burgers equation can be written for the deviation and, up to a global Galilean transformation, is scaling invariant. Therefore it makes sense to study the diffusive limiting behavior of the simple exclusion and try to prove that it is described by the viscous Burgers equation.

This represents a reasonably simple well defined situation where one can try to face the difficulty of computing the first correction to the local equilibrium. The main tool to do this is the Varadhan's non gradient method. This is in our opinion the rigorous way of computing the corrections to the local equilibrium. Given that, we suitably modify the local equilibrium to be compared with the true distribution and estimate the relative entropy between them using a modification of the entropy method. The result is proved in [18] and [32]. One of the consequences of the method is a rigorous proof of the Green-Kubo formula for the diffusion matrix, together with its variational characterization [31].

The simple exclusion is a model with just one conserved quantity and hence the diffusive limit is ruled by one macroscopic equation. The hydrodynamics of a fluid are typically a system of $d+2$ equations corresponding to the microscopic conservation of mass, momentum and energy. Therefore, to have a chance to converge to them, a microscopic model has to be much more complicated. Models with realistic hydrodynamical equations in the diffusive limit have been proposed in the framework of the *Cellular Automata* [22]. They are are chosen to be as deterministic as possible for numerical purposes, but from the theoretical point of view they are probably as difficult as the Hamiltonian particles system.

We introduce in [20] and [2] lattice models inspired by the cellular automata, but much more stochastic and hence better suited to our purposes. The models consist of several species of particles, labelled by finitely many vectors $v \in \mathbb{R}^d$ that will be interpreted as velocities of the particles. Each particle of a given species performs jumps as in the simple exclusion process, being careful not to jump where another particle of the same species is already present. The jumps in opposite directions are not equal and the resulting drift is assumed to be v if v is the label of the species. Moreover, at random Poisson times, particles change their velocities in consequence of collisions preserving mass, momentum and energy. The choice of the set of velocities is crucial for two main reasons: first, one has to be sure that the conserved quantities are exactly mass, momentum and energy. Second, the equations we are interested in are the incompressible Navier-Stokes equations and to get them we need a careful choice of the set of velocities. It is not obvious that one can satisfy such conditions. This is made possible by the presence of free parameters that can be suitably tuned. The model presented in [20] provides the incompressible Navier-Stokes equations, but no energy equation. In the model presented in [2] an equation for the energy is also obtained, so that the latter model is suitable to study thermal effects.

From the technical point of view the proof follows the same lines outlined for the simple exclusion, and will be briefly sketched in Sections 5 and 6. Also in this case the transport coefficients are given by a rigorous Green-Kubo formula and have a variational characterization.

2. BOLTZMANN EQUATION AND HYDRODYNAMIC LIMITS.

We consider a rarefied gas as made of particles which move freely for most of the time and sometimes are close enough to other particles to interact with them (*collisions*). The mean free path λ is the typical distance that a particle can cross without interacting with other particles. Clearly, if the density is finite, the mean free path is of the same order of the molecular size and therefore there is no *scale separation* between the microscopic scale and the kinetic scale. On the other hand, if the density is very small, so that the mean free path is much larger than the molecular size, the scale separation is ensured, so that the previous picture is correct and the Boltzmann equation is a reasonable approximation of the evolution of the system. The validity of the Boltzmann equation has been proved rigorously only one century after the Boltzmann work, for the *hard spheres* model in [33]. The main limitation of this result is the restriction to very short times. The global validity of the Boltzmann equation is still an open problem and actually a very difficult one. We do not enter here in the delicate discussion of such questions, referring the reader to [8] and references quoted therein. We rather assume the Boltzmann equation as the basic model for our discussion and in this section we disregard the underlying particle model. We will refer to the scale on which the Boltzmann equation is valid as the *kinetic* scale, to be compared to the much larger scale where the hydrodynamical equations hold, the *macroscopic* scale.

The kinetic description, intermediate between the microscopic and macroscopic ones, is specified by the density of particles with position r and velocity v, $f(r, v, t)$, in the six dimensional space, $r \in \Lambda$, $v \in \mathbb{R}^3$. The time evolution of f is given, for a gas, by the Boltzmann equation (BE) which has the form

$$\frac{\partial f(r, v, t)}{\partial t} + v \cdot \frac{\partial f}{\partial r} = Q(f, f) \tag{2.1}$$

where $Q(f, f)$ is a bilinear collision operator whose exact form depends on the interaction [8]. We have taken the mass of the particles to be unity.

Since the collision operator acts only on velocities, position and time being just parameters, we omit such dependence.

We have, after symmetrization,

$$Q(f,g)(v) = \frac{1}{2}\int_{I\!R^3} dv_* \int_{S_2} d\omega B(|v-v_*|,\omega) \atop [f'g'_* + f'_*g' - fg_* - f_*g], \quad (2.2)$$

where S_2 is the $2d$-sphere in $I\!R^3$, $d\omega$ is the surface measure on it,

$$f' = f(v'), \quad f'_* = f(v'_*), \quad f_* = f(v_*), \quad f = f(v).$$

The vectors v, v_* are the outgoing velocities of a binary elastic collision between two particles with incoming velocities v' and v'_*:

$$\begin{aligned} v' &= v - \omega[\omega\cdot(v-v_*)], \\ v'_* &= v_* + \omega[\omega\cdot(v-v_*)] \end{aligned}. \quad (2.3)$$

Finally, $B(|v-v_*|,\omega)$ is the differential cross section of the collision, depending on the interaction law. For hard spheres it is given by

$$B(|v-v_*|,\omega) = \frac{1}{2}|(v-v_*)\cdot\omega|, \quad (2.4)$$

Let f, g and h be functions of v such that the integrals below are well definite. Then
(i)

$$\begin{aligned} \int_{I\!R^3} dv Q(f,g)(v)h(v) = 0 \quad \text{iff} \\ h(v) = \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha(v) \end{aligned} \quad (2.5)$$

with c_α some real constants and

$$\chi_\alpha(v) = \begin{cases} 1 & \alpha = 0, \\ v_i & \alpha = i = 1,\ldots,3, \\ \frac{v^2}{2} & \alpha = 4. \end{cases} \quad (2.6)$$

The functions χ_α, $\alpha = 0,\ldots,4$ are called *collision invariants*. The vanishing of the integrals in (2.5) expresses the conservation of mass ($\alpha = 0$), momentum ($\alpha = 1,2,3$) and energy ($\alpha = 4$) during a collision.

(ii) If $f \geq 0$, then

$$\int_{I\!R^3} dv Q(f,f)(v) \log f(v) = N(f), \tag{2.7}$$

where the *entropy production* $N(f)$

$$\begin{aligned} N(f) = & \frac{1}{8} \int_{I\!R^3} dv \int_{I\!R^3} dv_* \int_{S_2} d\omega B(|v - v_*|, \omega) \\ & [f' f'_* - f f_*][\log f f_* - \log f' f'_*] \end{aligned} \tag{2.8}$$

is non positive and vanishes if and only if $\log f = \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha$.

(iii) Let $f(r, v, t)$ be a non negative, normalized function. Then it was shown by Boltzmann that

$$Q(f,f)(r,v,t) = 0 \tag{2.9}$$

if and only if

$$f(r,v,t) = M(\rho(r,t), u(r,t), T(r,t); v), \tag{2.10}$$

with

$$M(\rho, u, T; v) = \frac{\rho}{[2\pi T]^{3/2}} \mathrm{e}^{-\frac{(v-u)^2}{2T}} \tag{2.11}$$

where we have set Boltzmann constant equal to one.

The functions $M(\rho, u, T; v)$ satisfying (2.9) are called *local Maxwellians*. They play a crucial role in studying the hydrodynamical limit, because they are the *local equilibria* in the context of kinetic theory.

The local conservation laws for the mass, momentum and energy are an immediate consequence of the property (i). Define

$$\begin{aligned} \rho(r,t) &= \int f(r,v,t) dv, \\ \rho u(r,t) &= \int v f(r,v,t) dv, \\ \rho e(r,t) &= \int \frac{1}{2}(v-u)^2 f(r,v,t) dv. \end{aligned} \tag{2.12}$$

The density $\rho(r,t)$ and the internal energy $e(r,t)$ are non negative; $u(r,t)$ is the *stream velocity* vector. By multiplying (2.1) by χ_α, $\alpha = 0,\dots,4$ and integrating on v, by property (i) we get

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}[\rho u] = 0,\\
&\partial_t(\rho u_i) + \sum_{j=1}^{3} \partial_{r_j}\big(\rho u_i u_j + \Pi_{i,j}\big) = 0, \quad i = 1\dots,3\\
&\partial_t\Big[\rho\big(\frac{u^2}{2} + e\big)\Big] + \operatorname{div}\Big[\rho u\big(\frac{u^2}{2} + \frac{5}{3}e\big) + q\Big] = 0
\end{aligned} \tag{2.13}$$

where

$$\Pi_{i,j} = \int_{I\!R^3} dv f(r,v,t)(v_i - u_i(r,t))(v_j - u_j(r,t)) \tag{2.14}$$

is the *stress tensor* and

$$q = \int_{I\!R^3} dv f(r,v,t)(v - u(r,t))^2 (v - u(r,t)) \tag{2.15}$$

is the *heat current vector.*

Of course the above relations are *not* the hydrodynamical equations: the stress tensor and the heat current vector cannot be 'a priori' computed only in terms of ρ, u, e. The closure problem consists in finding such relations. The simplest case corresponds to assuming local equilibrium, i.e. that f is a local Maxwellian. In this case equations (2.13) are obviously closed and are just the Euler equations for an ideal gas, since for $f = M$ simple Gaussian integrations show that $\Pi_{i,j} = \rho T\ \delta_{i,j},\ e = 3T/2,\ q = 0.$ The assumption however is not correct in general because local Maxwellians are not solutions of the Boltzmann equation but when the parameters of M are independent of both r and t. In this case M is called the *global* Maxwellian. In fact an important point is missing in our discussion: the Boltzmann equation and the hydrodynamical equations hold on very different space scales, while (2.13) hold on the same scale of the Boltzmann equation. We introduce a scale separation parameter $\varepsilon = \lambda/L$ where L is the typical macroscopic length: roughly ε is the size of the mean free path in macroscopic units. We will show in the sequel that the assumption of local equilibrium is approximately correct for ε small and provides the hydrodynamical equations.

To explore the behavior of the system for ε small we choose as initial distribution a smooth function $f_0(\varepsilon r, v)$ which is slowly varying as function of r. The variable $x = \varepsilon r$ represents the macroscopic position. We assume for simplicity periodic boundary conditions, that is r varies in the torus of size ε^{-1}, $\boldsymbol{T}_\varepsilon = [0, \varepsilon^{-1}]^3$ so that correspondingly x varies in the unit torus $\boldsymbol{T} = [0,1]^3$: ε is thus just the inverse length of the box expressed in microscopic units. Since the initial datum is almost constant on the microscopic scale, the time evolution given by the Boltzmann equation (2.1) will not depart significantly from the uniform state before sufficiently long times. Just how long a time is required can be estimated from the fact that to depart from spatial uniformity a particle of the gas, moving with finite velocity v, has to cover a macroscopic distance to detect the variations of the initial datum. This means that it needs a time of order ε^{-1}. Therefore, we shall look at the solution of (2.1) at a time $\varepsilon^{-1}t$: we set

$$f^\varepsilon(x, v, t) = f(\varepsilon^{-1}x, v, \varepsilon^{-1}t) \qquad (2.16)$$

with $f(r, v, t)$ solving (2.1) with initial datum $f_0(\varepsilon r, v)$. We will refer to the space-time scaling used in (2.16) as *Euler* scaling.

The rescaled distribution $f^\varepsilon(x, v, t)$ has to solve the *rescaled* BE:

$$\frac{\partial f^\varepsilon}{\partial t} + v \cdot \frac{\partial f^\varepsilon}{\partial x} = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \qquad (2.17)$$

with initial datum $f_0(x, v)$. We expect to recover the hydrodynamical equations in the limit $\varepsilon \to 0$. Formally, it is clear that in this limit the right hand side of (2.17) would become singular. The only possibility to avoid this singularity is that

$$\lim_{\varepsilon \to 0} Q(f^\varepsilon, f^\varepsilon) = 0. \qquad (2.18)$$

Condition (2.9),(2.10) then forces f^ε to converge to a local Maxwellian. As remarked before, this, together with the local conservation laws produces a set of closed equations for the hydrodynamical variables, which are the Euler equations for the ideal fluid.

The picture is then the following: If we consider a small macroscopic region $\Delta = \delta\varepsilon^{-1}$ in Λ centered in x, the system, when observed on a time scale of order ε^{-1}, will thermalize so that its distribution will be described by a Maxwellian, whose parameters will evolve according to the Euler equations. This means that we can divide the observables into two classes, according to their relaxation times: the fast observables, which relax to the equilibrium values on a time scale much shorter then ε^{-1} and will not have any effect on the hydrodynamical scales and the slow observables which are locally conserved by the dynamics and, for this reason, need much longer times to relax.

The quantitative analysis of this asymptotic behavior goes back to Hilbert [25], who proposed the famous *Hilbert expansion* as an example of his theory of linear transformations. The starting point is to look for the solution f^ε as a power series in ε:

$$f^\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k f_k, \tag{2.19}$$

and try to find the functions f_k, $k \geq 0$ by imposing that (2.17) be satisfied at each order in ε. By plugging the series (2.19) in (2.17) and equating terms of the same order in ε we get the following conditions:

$$\begin{aligned}
\varepsilon^{-1}: \qquad & Q(f_0, f_0) = 0, \\
\varepsilon^{0}: \qquad & 2Q(f_0, f_1) = \frac{\partial f_0}{\partial t} + v \cdot \frac{\partial f_0}{\partial x}, \\
& \quad \dots \\
\varepsilon^{k}: \qquad & 2Q(f_0, f_k) = S_k(f_0, f_1, \dots, f_{k-1}) \\
& \quad \dots,
\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}
S_k(f_0, f_1, \dots, f_{k-1}) = {} & \frac{\partial f_{k-1}}{\partial t} + v \cdot \frac{\partial f_{k-1}}{\partial x} \\
& - \sum_{\substack{(h,h'):\ h,h' \geq 1 \\ h+h'=k}} Q(f_h, f_{h'}) \cdot
\end{aligned} \tag{2.21}$$

The order -1 in ε in (2.20) implies, using (2.9), (2.10), (2.11) that f_0 is a local Maxwellian $M = M(\rho, u, T; v)$ with the functions ρ, u, T not yet determined. The other equations (2.20) can be written in the form

$$\mathcal{L} f_k = S_k(f_0, f_1, \ldots, f_{k-1}), \quad k \geq 1. \tag{2.22}$$

where $\mathcal{L}$ is the *linearized Boltzmann operator*

$$\mathcal{L} g = 2Q(M, g). \tag{2.23}$$

Before concluding the discussion of the expansion we state some properties of $\mathcal{L}$ we will extensively use below. We refer to [24] for the proofs.

It is convenient to introduce also

$$Lg = \frac{1}{\sqrt{M}} \mathcal{L}(\sqrt{M} g) = \frac{2}{\sqrt{M}} Q(M, \sqrt{M} g) \tag{2.24}$$

Simple calculations show that $\mathcal{L}$ and L can be decomposed as

$$\mathcal{L} = -\nu(v) + \mathcal{K}, \quad L = -\nu(v) + K, \tag{2.25}$$

where $\nu(v) \geq \nu_0 > 0$ is a smooth function growing at large velocities as the cross section; K is an integral operator which is compact on the space of square integrable functions $\mathcal{H} = L_2(\mathbb{R}^3)$. Therefore the operator L is well defined in the domain $\mathcal{D} \subset \mathcal{H}$ of the functions $g(v)$ such that $\nu g \in \mathcal{H}$. Some properties of L are:

i) L is symmetric in $\mathcal{H}$:

$$(g, Lf) = (f, Lg), \quad f, g \in \mathcal{D}; \tag{2.26}$$

ii) L is non positive in $\mathcal{H}$:

$$(f, Lf) \leq 0 \quad f \in \mathcal{D}; \tag{2.27}$$

iii) Let

$$\psi_\alpha = \sqrt{M} \chi_\alpha, \quad \alpha = 0, \ldots, 4, \tag{2.28}$$

$\mathcal{P}$ be the projector on the space $\mathcal{I}$ spanned by $\{\psi_\alpha, \alpha = 0, \ldots, 4\}$ and $\mathcal{P}^\perp$ the projector on $\mathcal{I}^\perp$, the orthogonal complement of $\mathcal{I}$. Then

$$\text{Null } L = \mathcal{I}, \quad \text{Range } L \subset \mathcal{I}^\perp. \tag{2.29}$$

iv) There is $\mu > 0$ such that, if $f \in \mathcal{P}^{\perp}\mathcal{D}$, then

$$(f, Lf) \leq -\mu(f, \nu f) \qquad (2.30).$$

We now continue to construct the terms of the expansion by discussing the solution of the equation

$$\mathcal{L}f = g. \qquad (2.31)$$

With the notation $\tilde{f} = fM^{-1/2}$, $\tilde{g} = gM^{-1/2}$, (2.31) is equivalent to

$$L\tilde{f} = \tilde{g} \qquad (2.32)$$

Since Null L is not trivial, L is not invertible. Only its restriction to $\mathcal{I}^{\perp}$ can be inverted. Hence, if $\tilde{g}$ is in $\mathcal{I}^{\perp}$, the space perpendicular to the invariant subspace $\mathcal{I}$, it is possible to find a solution and, by the compactness of K and the application of the Fredholm alternative, we can conclude that there exists one and only one solution of (2.32) in $\mathcal{I}^{\perp}$. The condition on $\tilde{g}$ is then

$$\mathcal{P}\tilde{g} = 0 \equiv (\tilde{g}, \psi_\alpha) = 0, \quad \alpha = 0, \ldots, 4. \qquad (2.33)$$

and the solution to (2.32) can be denoted as $L^{-1}\tilde{g}$. The general solution to (2.32) is therefore

$$\tilde{f} = L^{-1}\tilde{g} + \sum_{\alpha=0}^{4} c_\alpha \psi_\alpha. \qquad (2.34)$$

Going back to (2.31), we can conclude that, provided that g satisfies

$$\int_{\mathbb{R}^3} d\underline{v}\chi_\alpha(\underline{v})g(\underline{v}) = 0, \quad \alpha = 0, \ldots, 4. \qquad (2.35)$$

and $gM^{-1/2} \in L_2(\mathbb{R}^3)$, the general solution of (2.31) is given by

$$f = \mathcal{L}^{-1}g + \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha M, \qquad (2.36)$$

where we used the (slightly inaccurate) notation

$$\mathcal{L}^{-1}g = M^{1/2}L^{-1}(gM^{-1/2}). \qquad (2.37)$$

We now return to the study of (2.22). Let us consider first $k = 1$. The solvability condition in this case is:

$$\int_{I\!R^3} dv \chi_\alpha(v) D_t M(\rho(x,t), u(x,t), T(x,t); v) = 0,$$
$$\alpha = 0, \dots, 4. \tag{2.38}$$

which coincides with (2.13) when putting $f = M$. Therefore we can conclude that the solvability condition (2.38) is satisfied if and only if the parameters of the Maxwellian satisfy the equations

$$\begin{aligned} &\partial_t \rho + \operatorname{div}[\rho u] = 0, \\ &\rho \partial_t u + \rho(u \cdot \nabla_x) u = -\nabla_x P \\ &\rho \partial_t e + \rho(u \cdot \nabla_x) e + P \operatorname{div} u = 0, \end{aligned} \tag{2.39}$$

where

$$e = \frac{3}{2} T, \quad P = \rho T. \tag{2.40}$$

are the equations of state for the pressure and internal energy of the *ideal gas*. It should be noted that, since we are considering a situation in which the average distance between the particles is much larger than their interaction range these are indeed the appropriate equations of state.

The previous discussion shows that to the lowest order in ε the solution of the rescaled Boltzmann equation is a local Maxwellian whose parameters solve the Euler equations of the ideal fluid. The method allows to compute corrections of arbitrary order. In fact, once the solvability condition (2.38) is satisfied, f_1 is given by

$$f_1 = \mathcal{L}^{-1}[D_t M] + \sum_{\alpha=0}^{4} c_\alpha^{(1)} \chi_\alpha M, \tag{2.41}$$

with $c_\alpha^{(1)}(x,t)$, $\alpha = 0, \dots 4$, arbitrary functions. The presence of arbitrary functions permits to continue the procedure and determine f_k for $k > 1$. Suppose that at some step k the function $f_1, \dots f_{k-1}$ are determined uniquely and f_k is determined up to arbitrary functions $c_\alpha^{(k)}(x,t)$, $\alpha = 0, \dots, 4$:

$$f_k = \bar{f}_k + \sum_{\alpha=0}^{4} c_\alpha^{(k)} \chi_\alpha M. \tag{2.42}$$

In order to determine f_{k+1} we have to solve

$$\mathcal{L} f_{k+1} = S_{k+1}(M, f_1, \ldots, f_k) \tag{2.43}$$

The solvability condition

$$\int_{\mathbb{R}^3} dv \chi_\alpha(v) S_{k+1}(v), \quad \alpha = 0, \ldots, 4 \tag{2.44}$$

can be satisfied taking advantage of the arbitrary functions $c_\alpha^{(k)}(x,t)$: in fact (2.44) is a set of five linear PDE's in the unknown functions $c_\alpha^{(k)}(x,t)$, which fully determine such functions once initial conditions are provided. Therefore f_{k+1} can be determined by (2.43), up to arbitrary functions $c_\alpha^{(k+1)}$ and the procedure can be iterated indefinitely.

In conclusion all the coefficients f_k are uniquely determined by above procedure. Unfortunately there is no indication that the power series (2.19) has a finite radius of convergence, so that the previous argument is only formal.

On the other hand, one can hope that finite truncations of the series are good approximations of the solution, if any. In order to see this, one fixes $N > 0$ and looks for a solution of (2.17) of the form

$$f^\varepsilon = \sum_{k=0}^{N} \varepsilon^k f_k + \varepsilon^m R, \tag{2.45}$$

with f_k, $k = 0, \ldots, N$ determined by the previous procedure and R a suitable remainder function, depending on ε and N, to be chosen so that f^ε solves (2.17). Note that one would expect $m = N + 1$, but one can try to prove a weaker statement by choosing $m < N + 1$. Actually this is the case and the following theorem has been proved:

Theorem 2.1 *Assume that there exists a unique smooth solution in $H_s(\mathbb{T})$, with s sufficiently large. $(\rho(x,t),\ u(x,t),\ T(x,t))$ to the initial value problem associated to the Euler equations (2.40), in a finite time interval $[0, t_0]$. Denote by M_t the local Maxwellian with above parameters. Suppose that*

$$\|f_0 - M_0\| < \delta \tag{2.46}$$

for some $\delta > 0$ sufficiently small. Then there is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, there exists a smooth *solution $f^\varepsilon(t)$ to the rescaled Boltzmann equation, with*

$$\sup_{t\in(0,t_0]} \|f^\varepsilon(t) - M_t\| \le C\varepsilon \tag{2.47}$$

for some $C > 0$. The norm is defined as

$$\|g\| = \sup_{v\in\mathbb{R}^3} (1+v^2)^k \|g(\,\cdot\,,v)\|_{H_s(\mathbb{T})}. \tag{2.48}$$

with $k, s > 3/2$, $\|\cdot\|_{H_s(\mathbb{T})}$ being the Sobolev norm of order s on $\mathbb{T}$)

Theorem 2.1 has been proved in the case of 1-dimensional space variations and $\delta = 0$ by Caflisch in [6], who used the truncated expansion with $N = 6$, $m = 3$. Note that such a choice of N and m is just the simplest possible choice and one could choose other pairs of values. Extensions to higher space dimensions and $\delta > 0$ have been given in [28]. The supremum on time in (2.47) is restricted to $t > 0$. This is due to the fact that δ can be much bigger than ε. Even in this case, the solution at positive macroscopic times is close to the local Maxwellian up to $O(\varepsilon)$, because it is attracted to a neighborhood of the local Maxwellian on a time scale much shorter than the hydrodynamical one (*initial layer*).

Other results on the convergence of the solution of the rescaled Boltzmann equation to the Euler equations are due to [35] and [42]. These results are based on the argument (2.18) and do not use the Hilbert expansion. They hold only for short times. In fact, it should be stressed that the locality in time of Theorem 2.1 is essentially of hydrodynamical type, in the sense that the solution of the Euler equation could become singular after t_0. If a global solution of the Euler equations is given then Theorem 2.1 holds with t_0 arbitrarily large (and the constant C possibly diverging for $t_0 \to +\infty$). On the contrary, the locality of the results in [35] and [42] depend on the time necessary to the solution of the Boltzmann equation to break down, if any.

Navier-Stokes limit

This derivation of the Euler equations in the above hydrodynamical (Euler) scaling limit is consistent with (indeed made possible by) the fact that the Euler equations are themselves invariant under uniform space and time scaling. Unfortunately there is no such scale invariance (and thus no such scaling limit) for the Navier-Stokes equations (NSE). The NSE are usually deduced, via the Chapman-Enskog expansion (see [7]), as corrections to the Euler equations on the Euler time scale ε^{-1} with viscosity coefficients and thermal conductivity of order ε. To compute these corrections it is enough to look at the first order term in the Hilbert expansion, f_1. We remind that by (2.20) f_1 is determined by solving the equation

$$\mathcal{L}f_1 = \frac{\partial M}{\partial t} + v \cdot \frac{\partial M}{\partial x}$$

While the explicit form of $\mathcal{L}^{-1}[D_t M]$ is only available for particular molecular interactions (Maxwell molecules) a general expression for f_1 can be obtained (see [7]): let $\widetilde{v} = v - u$ and

$$\begin{aligned} A_{i,j}(v) =& \widetilde{v}_i \widetilde{v}_j - \frac{\widetilde{v}^2}{3}\delta_{i,j}, \quad B_i(v) = (\frac{\widetilde{v}^2}{2} - \frac{5}{2}T)\widetilde{v}_i, \\ & i,j = 1,\dots,3. \end{aligned} \tag{2.49}$$

It is immediate to verify that, if we define $\widetilde{\mathcal{P}}^{\perp} g = M^{1/2}\mathcal{P}^{\perp}[M^{-1/2}g]$, where $\mathcal{P}^{\perp}$ is the projection into the space $\mathcal{I}^{\perp}$, then

$$\widetilde{\mathcal{P}}^{\perp}[D_t M] = M\Big[\sum_{i,j=1}^{3} A_{i,j}\frac{\partial u_i}{\partial x_j} + \sum_{i=1}^{3} B_i \frac{\partial T}{\partial x_i}\Big]. \tag{2.50}$$

Moreover, there are two non negative functions, depending on $|\widetilde{v}|$, Ψ_1 and Ψ_2 such that

$$\begin{aligned} \mathcal{L}^{-1}\big[A_{i,j}M\big] &= -\Psi_1 A_{i,j}, \\ \mathcal{L}^{-1}\big[B_i M\big] &= -\Psi_2 B_i](v). \end{aligned} \tag{2.51}$$

Therefore

$$\mathcal{L}^{-1}[D_t M] = -\Psi_1 \sum_{i,j=1}^{3} A_{i,j}\frac{\partial u_i}{\partial x_j} - \Psi_2 \sum_{i=1}^{3} B_i \frac{\partial T}{\partial x_i}. \tag{2.52}$$

This is the part of f_1 in $\mathcal{I}^{\perp}$. The part of f_1 in $\mathcal{I}$ represents the first order correction to the hydrodynamical quantities. The Chapmann-Enskog expansion is essentially a suitable resummation of the Hilbert expansion and provides a closed set of equations, accurate to first order in f_1, for the hydrodynamical fields:

$$
\begin{aligned}
&\partial_t\rho + \operatorname{div}[\rho u] = 0,\\
&\rho\partial_t u + \rho(u\cdot\nabla_x)u =\\
&-\nabla_x P + \nabla_x\cdot(\mu_\varepsilon\nabla_x u) + \nabla_x(\zeta_\varepsilon \operatorname{div} u),\\
&\rho\partial_t e + \rho(u\cdot\nabla_x)e + P\operatorname{div} u =\\
&\nabla_x\cdot(\kappa_\varepsilon\nabla_x T) + \mu_\varepsilon(\nabla_x u)^2 + \zeta_\varepsilon(\operatorname{div} u)^2,
\end{aligned}
\tag{2.53}
$$

with $\mu_\varepsilon = \varepsilon\mu$, $\zeta_\varepsilon = \varepsilon\zeta$, $\kappa_\varepsilon = \varepsilon\kappa$ and

$$
\begin{aligned}
\mu &= \int_{I\!R^3} dv[A_{1,2}(v)]^2\Psi_1(|v|),\\
\zeta &= \int_{I\!R^3} dv[A_{1,1}(v)]^2\Psi_1(|v|),\\
\kappa &= \int_{I\!R^3} dv[B_1(v)]^2\Psi_2(|v|).
\end{aligned}
\tag{2.54}
$$

The coefficients μ, ζ and κ are called *transport coefficients* and, in particular, μ is the *shear viscosity*, ζ the *volume viscosity* and κ the *heat conduction coefficient*. The above formulas are similar to the Green-Kubo formulas [39] for the transport coefficients in statistical mechanics, which reduce to the former in the limit. Equations (2.53), with $\varepsilon = 1$ are the well known Navier-Stokes equations for a viscous fluid.

The derivation of the compressible Navier-Stokes equations based on the Chapmann-Enskog procedure is not satisfactory: different resummation procedures could give in principle different corrections (see the discussion for a hard rods system in [40]). Terms of order ε cannot be distinguished on the Euler time scale, because the corresponding solutions tend to coincide to lowest order in the limit $\varepsilon \to 0$. On the other hand the viscosity produces significant effects on very long times, namely times of order ε^{-1} with respect the Euler time scale (hence of order ε^{-2} with respect to the microscopic scale), so that the first order corrections become important on these times and give rise to distinctive behavior. Not surprisingly there are many difficulties in finding these corrections which would be valid for such long times: if the Euler solution is unstable, a perturbation of order ε could produce explosive behaviors at such long times. Even assuming enough stability properties to control the explosion of the corrections, they could however be sensible to the initial conditions.

If we consider from the beginning a longer time scale $\varepsilon^{-2}t$, instead of $\varepsilon^{-1}t$, that makes the time derivatives of the same order as the viscous terms, but in this way the nonlinear terms involving the first order space derivatives are of order ε^{-1} while the other terms are of order 1. On the other hand if one also scales the macroscopic velocity u, namely one considers u's very small compared to macroscopic velocities, for example the sound speed c, in such a way that the Mach number $Ma = u/c$ is of order ε, then there is a scaling which leaves invariant a "subset" of the Navier Stokes equations, the incompressible Navier-Stokes equations (INS)

$$\begin{aligned} &\mathrm{div} u = 0, \\ &\partial_t u + (u \cdot \nabla_x) u = F - \nabla_x p + \eta \Delta u, \end{aligned} \tag{2.55}$$

with η the *kinematic viscosity coefficient.* The pressure p appearing in (2.55) is no more given by a state equation, but is itself an unknown of the problem. We notice that (2.55) is invariant under the scaling

$$x' = \varepsilon^{-1} x, \quad t' = \varepsilon^{-2} t, \quad u' = \varepsilon u, \quad p' = \varepsilon^2 p. \tag{2.56}$$

Based on these remarks we investigate the asymptotic behavior of

$$f^\varepsilon(x, v, t) = f(\varepsilon^{-1} x, v, \varepsilon^{-2} t), \tag{2.57}$$

solution of

$$\frac{\partial f^\varepsilon}{\partial t} + \frac{1}{\varepsilon} v \cdot \frac{\partial f^\varepsilon}{\partial x} = \frac{1}{\varepsilon^2} Q(f^\varepsilon, f^\varepsilon). \tag{2.58}$$

We consider the following initial condition:

$$f_0^\varepsilon(x, v) = m(v) + \varepsilon \Phi_0(x, v), \quad m(v) = M(1, 0, 1; v). \tag{2.59}$$

This means that at the lowest order the initial distribution is a global Maxwellian with vanishing average velocity. Therefore

$$u_0(x) = \varepsilon \int_{I\!R^3} dv \Phi_0(x, v) v, \tag{2.60}$$

so the low Mach number assumption is fulfilled. We are also assuming constant initial density and temperature both set equal to one. We remark that the formal part of the argument below can be worked out even in the case of non constant T and ρ [9], where new interesting phenomena arise, e.g. mass flows induced by temperature gradients, which are not contained in the INS.

Given the initial condition (2.59) we look for a solution of (2.58) with the same property: we set

$$f^\varepsilon(x,v,t) = m(v) + \varepsilon\Phi^\varepsilon(x,v,t). \tag{2.61}$$

This assumption is consistent with (2.58) provided that Φ^ε solves the equation

$$\frac{\partial\Phi^\varepsilon}{\partial t} + \frac{1}{\varepsilon}v\cdot\frac{\partial\Phi^\varepsilon}{\partial x} = \frac{1}{\varepsilon^2}\mathcal{L}\Phi^\varepsilon + \frac{1}{\varepsilon}Q(\Phi^\varepsilon,\Phi^\varepsilon). \tag{2.62}$$

We try to solve this equation using a kind of Hilbert expansion along the same lines presented for the Euler limit. We look for a solution of (2.62) in the form

$$\Phi^\varepsilon = \sum_{k=0}^{\infty}\varepsilon^k\phi_k \tag{2.63}$$

and determine the unknown coefficients ϕ_k, $k = 0,\ldots,$ by imposing equality of terms of the same order in ε. By plugging (2.63) into (2.62) we get the conditions:

$$\begin{aligned}
\varepsilon^{-2}: &\qquad \mathcal{L}\phi_0 = 0,\\
\varepsilon^{-1}: &\qquad \mathcal{L}\phi_1 = v\cdot\nabla_x\phi_0 - Q(\phi_0,\phi_0)\\
\varepsilon^{0}: &\qquad \mathcal{L}\phi_2 = \partial_t\phi_0 + v\cdot\nabla_x\phi_1 - 2Q(\phi_0,\phi_1),\\
&\qquad \ldots\\
\varepsilon^{k}: &\qquad \mathcal{L}\phi_{k+2} = Z_k(\phi_0,\phi_1,\ldots,\phi_{k+1})\\
&\qquad \ldots,
\end{aligned} \tag{2.64}$$

The first condition and (2.5) imply

$$\begin{aligned}\phi_0(x,v,t) = m(v)\Big[&\rho(x,t) + v\cdot u(x,t)\\ &+ \big(\frac{v^2-3}{2}\big)\vartheta(x,t)\Big],\end{aligned} \tag{2.65}$$

where we have denoted by ρ and ϑ the first order corrections to the constant density and temperature and by u the first non vanishing contribution to the stream velocity vector, which is of order ε by construction. This amounts to say that the correction of order ε in f^ε belongs to the null space of $\mathcal{L}$, the linearized Boltzmann operator around the global Maxwellian.

The other equations in (2.64) are again in the form (2.31). Therefore, in order to solve them, we have to check that the right hand sides satisfy the solvability condition of orthogonality to the null space of $\mathcal{L}$. To find ϕ_1 we need

$$\int_{I\!R^3} dv \chi_\alpha(v) v \cdot \nabla_x \phi_0(x,v,t) = 0, \quad \alpha = 0,\dots,2.4. \tag{2.66}$$

It is easy to check, by explicit Gaussian integrations that the above conditions are satisfied if and only if

$$\mathrm{div} u = 0, \quad \nabla_x(\rho + \vartheta) = 0. \tag{2.67}$$

The first equality in (2.67) is the *incompressibility* condition. The second one, which ensures the constancy of the pressure up to the first order, is called the Boussinesq condition (see [5]). Once we choose u, ρ and ϑ so that (2.67) is satisfied, the second equation of (2.64) can be solved and we get

$$\phi_1 = \mathcal{L}^{-1}[v \cdot \nabla_x \phi_0] - \mathcal{L}^{-1}[Q(\phi_0,\phi_0)] + \chi_1, \tag{2.68}$$

with $\chi_1 \in \mathrm{Null}\ \mathcal{L}$, and hence of the form

$$\chi_1(x.v,t) = m(v)\Big[\rho_1(x,t) + v \cdot u_1(x,t) + \\ (\frac{v^2-3}{2})\vartheta_1(x,t)\Big]. \tag{2.69}$$

The quantity $\mathcal{L}^{-1}[v\cdot\nabla_x\phi_0]$ can be computed using the same arguments employed to get (2.52) and the result is

$$\mathcal{L}^{-1}[v \cdot \nabla_x \phi_0] = -\Psi_1 \sum_{i,j=1}^{3} A_{i,j}\frac{\partial u_i}{\partial x_j} - \Psi_2 \sum_{i=1}^{3} B_i \frac{\partial \vartheta}{\partial x_i}. \tag{2.70}$$

To evaluate the second term in (2.68) we note that, by simple calculations one can get

$$
\begin{aligned}
&-\mathcal{L}^{-1}[Q(\phi_0,\phi_0)] = \frac{1}{2}\widetilde{\mathcal{P}}^{\perp}\left(\frac{\phi_0^2}{m}\right) \\
&= \frac{1}{2}m\sum_{i,j=1}^{3} A_{i,j}u_iu_j + m\vartheta\sum_{i=1}^{3} B_iu_i \qquad (2.71)\\
&+ \frac{1}{2}\vartheta^2\widetilde{\mathcal{P}}^{\perp}\Big[\big(\frac{v^2-3}{2}\big)^2 m\Big],
\end{aligned}
$$

ϕ_1 is thus known up to χ_1 which will be determined by the solvability condition for the third of (2.64). The expression just obtained for ϕ_1 and the evaluation of some Gaussian integrals provides the explicit solvability condition:

$$
\begin{aligned}
&\partial_t\rho + \text{ div } u_1 = 0,\\
&\partial_t u + (u\cdot\nabla_x)u = -\nabla_x[\frac{u^2}{2} + \rho\vartheta + \rho_1 + \vartheta_1] + \eta\Delta u,\\
&\frac{3}{2}\partial_t\vartheta + \text{ div } u_1 + (u\cdot\nabla_x)\vartheta = \bar{\kappa}\Delta\vartheta,
\end{aligned} \tag{2.72}
$$

where η and $\bar{\kappa}$ are the the shear viscosity coefficient and the heat conduction coefficient given by (2.54), evaluated at temperature $T=1$. We put

$$
p = \rho\vartheta + \rho_1 + \vartheta_1 + \frac{u^2}{2} \tag{2.73}
$$

and replace $\text{div}u_1$ in the third equation with $-\partial_t\rho = \partial_t\vartheta$. Therefore we get the following system in the unknowns u, ϑ and p:

$$
\begin{aligned}
&\text{div}u = 0,\\
&\partial_t u + (u\cdot\nabla_x)u = F - \nabla_x p + \eta\Delta u,\\
&\frac{5}{2}\partial_t\vartheta + (u\cdot\nabla_x)\vartheta = \bar{\kappa}\Delta\vartheta,
\end{aligned} \tag{2.74}
$$

The first two equations are just the INS equations (2.55) for the unknowns u and p. As in the derivation of the Euler equations they arise as solvability conditions for the coefficients of the power series expansion (2.63). The third of (2.74) is the heat equation with a convective term due to the streaming velocity: its solution does not influence u.

Suppose that, given appropriate initial conditions, a unique solution can be found for the system (2.74), in some time interval $(0, t_0]$. Then, by the Boussinesq condition, ρ is determined up to an additive constant and hence the first term in the expansion ϕ_0 is completely determined. The second term, ϕ_1, is determined up to χ_1, but with ρ_1, u_1 and ϑ_1 not completely arbitrary because div u_1 is prescribed by the first of (2.72) and the sum $\rho_1 + \vartheta_1$ is prescribed by (2.73). The procedure can be continued to determine the next term f_2, by taking advantage of the partial arbitrariness of χ_1 to satisfy the solvability condition for ϕ_2. In fact we are now in the same position as we were in the previous step. The only difference is that the equations (2.74) for ρ, u and ϑ are non linear, while those for ρ_1, u_1 and ϑ_1 are linear. The argument goes on as for the Euler case and the coefficients ϕ_k can be determined for any k.

As in the Euler case, nothing is known about the convergence of the series. Also in this case it is possible to introduce a suitable truncation of the series and try to estimate the remainders. To do this, fix $N > 0$, and put

$$\Phi^\varepsilon = \sum_{k=0}^{N} \varepsilon^k \phi_k + \varepsilon^m R, \tag{2.75}$$

with ϕ_k, $k = 0, \dots, N$ computed according to the previous procedure and R, depending on ε, to be chosen so that Φ^ε solves (2.62). For sake of definiteness, assume $N = 6$ and $m = 3$. The existence of the function R is the main ingredient to prove the following

Theorem 2.2 *Assume that the initial value problem for the system (2.74) has a unique, sufficiently smooth solution in a time interval* $(0, t_0]$. *Then there is an* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$, *there exists a smooth solution* $\Phi^\varepsilon(t)$ *to (2.62), having the property that, for some* $C > 0$,

$$\sup_{t \in (0, t_0]} \|\Phi^\varepsilon(t) - \phi_0(t)\| \leq C\varepsilon \tag{2.76}$$

Theorem 2.2 has been proved in [9]. We remark that the existence of a solution to the hydrodynamical equations is among the hypotheses of the theorem, and hence the result inherits, as in the Euler case, the locality in time of the hydrodynamical problem. A different approach has been considered in [1], where the aim is to get existence of global weak solutions to the hydrodynamical equations as a consequence of the existence of weak solutions of the rescaled Boltzmann equation, proved in [13]. This interesting program has turned out to be very hard to be completed. In [16] it has been proved the analogous of Theorem 2.2 (also for stationary solutions) in presence of a boundary and a constant force field. Other results concerning stationary situations has been given in [14] and [15], where suitable scaling limits bringing to compressible stationary Navier-Stokes equations have been proved.

3. HAMILTONIAN SYSTEMS: EURISTICS.

We consider a system of N identical particles of unit mass in a cube of size ε^{-1} in $\mathbb{R}^d$, with periodic b.c., that we denote by $\mathbb{T}_\varepsilon^d$, interacting through a pair central potential V of finite range, R. To have a finite density we choose $N = \varepsilon^{-d}$. Let ξ_i and v_i denote the position and velocity of the i-th particle, $i = 1 \dots, N$, τ the time and $\big(\xi(\tau), v(\tau)\big) = \big(\xi_1(\tau), v_1(\tau), \dots, \xi_N(\tau), v_N(\tau)\big) \in (\mathbb{T}_\varepsilon^d \times \mathbb{R}^d)^N$ a solution of the Newton equation for this system. In order to exploit the scale separation we introduce the rescaled trajectory $\big(x^\varepsilon(t), v^\varepsilon(t)\big) \in \Gamma_N = (\mathbb{T}_1^d \times \mathbb{R}^d)^N$, with $x_i^\varepsilon(t) = \varepsilon \xi_i(\varepsilon^{-1}t)$ and $v_i^\varepsilon(t) = v_i(\varepsilon^{-1}t)$, which are solutions to the rescaled Newton equations

$$
\begin{aligned}
&\frac{dx_i}{dt}(t) = v_i(t) \\
&\frac{dv_i}{dt}(t) = -\varepsilon^{-1} \sum_{j \neq i} \nabla V(\varepsilon^{-1}|x_i - x_j|)
\end{aligned}
\tag{3.1}
$$

for $i = 1, \dots, N$.

The total number of particles, the d components of the total momentum and the total energy are conserved by the dynamics. We define the corresponding empirical fields: we use the notation

$$
\begin{aligned}
&z_i^0 = 1; \quad z_i^\alpha = v_i^\alpha, \quad \alpha = 1, \dots, d; \\
&z_i^{d+1} = \frac{1}{2}[v_i^2 + \sum_{i \neq j} V(\varepsilon^{-1}|x_i - x_j|)],
\end{aligned}
$$

for $i = 1, \dots, N$ and, for $\mu = 0, \dots, d+1$

$$z^\mu(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x_i - x) z_i^\mu \tag{3.2}$$

denote the *empirical density* ($\mu = 0$), the *empirical velocity field density* ($\mu = 1, \dots, d$) and the *empirical energy density* ($\mu = d+1$).

They are measure-valued fields, depending on the configuration $(x, v) \in \Gamma_N$, whose integrals over a macroscopic region are equal to the average number of particles, momentum, energy in that region. The integrals on $\mathbb{T}_1^d$ are (up to a factor N) the total mass, momentum and energy.

The reason to introduce above empirical fields is that they satisfy formally a set of *local conservation laws* of the form

$$\frac{\partial}{\partial t} z^\mu + \sum_{k=1}^{d} \frac{\partial}{\partial x_k} W^{\mu,k} = 0 \tag{3.3}$$

that are readily obtained by computing the empirical fields on a suitably smooth test function f, differentiating with respect to t and using the Newton equations:

$$\frac{d}{dt} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial x_i^\alpha}(x_i) v_i^\alpha \tag{3.4}$$

$$\begin{aligned} \frac{d}{dt} \frac{1}{N} \sum_{i=1}^{N} f(x_i) v_i^\beta = \frac{1}{N} \sum_{i=1}^{N} \Big\{ & \frac{\partial f}{\partial x_i^\alpha}(x_i) v_i^\alpha v_i^\beta \\ & - \varepsilon^{-1} \sum_{j \neq i} \nabla_\beta V(\varepsilon^{-1}|x_i - x_j|) f(x_i) \Big\} \end{aligned} \tag{3.5}$$

$$\begin{aligned} \frac{d}{dt} \frac{1}{N} \sum_{i=1}^{N} f(x_i) z_i^{d+1} = \frac{1}{N} \sum_{i=1}^{N} \Big\{ & \frac{\partial f}{\partial x_i^\alpha}(x_i) v_i^\alpha z_i^{d+1} \\ - \frac{1}{2} \varepsilon^{-1} \sum_{i \neq j} \nabla_\beta V(\varepsilon^{-1}|x_i - x_j|) & [f(x_i) - f(x_j)] \Big\} \end{aligned} \tag{3.6}$$

Here $\nabla_\beta V(\xi) = \partial V(\xi)/\partial \xi_\beta$ and we used the symmetry properties of the potential to get the last term in

(3.6). In order to evaluate the last terms in (3.5) and (3.6) we take advantage of the fact that the two factors vary on different space scales. It is convenient to return to the microscopic coordinates ξ_i and note that with $\xi_i = \varepsilon^{-1} x_i$, we have

$$f(\varepsilon\xi_i) - f(\varepsilon\xi_j) = \sum_{\gamma=1}^{d} \frac{\partial f}{\partial x_i^\gamma}(x_i)[\xi_i^\gamma - \xi_j^\gamma]$$

$$+ \frac{1}{2}\varepsilon^2 \sum_{\gamma,\nu=1}^{d} \frac{\partial^2 f}{\partial x_i^\gamma \partial x_i^\nu}(x_i)[\xi_i^\gamma - \xi_j^\gamma][\xi_i^\nu - \xi_j^\nu] + O(\varepsilon^3) \tag{3.7}$$

Due to the symmetry of the potential the last term of eq.(3.5) becomes

$$\frac{1}{2}\frac{1}{N}\sum_{i\neq j}\sum_{\gamma} \frac{\partial f}{\partial x_i^\gamma}(x_i)\Psi^{\beta\gamma}(\varepsilon^{-1}(x_i - x_j)) + O(\varepsilon^2) \tag{3.8}$$

with

$$\Psi^{\beta,\gamma}(\xi) = -\nabla_\beta V(\xi)\xi^\gamma \tag{3.9}$$

An analogous computation can be done for the energy equation. Therefore we obtain the following form for the currents $W^{\mu,k}$:

$$W^{\mu,k}(x) = w^{\mu,k}(x) + \vartheta^{\varepsilon}_{\mu,k}(x), \tag{3.10}$$

with

$$w^{\mu,k}(x) = \frac{1}{N}\sum_{i=1}^{N} w_i^{\mu,k}\delta(x - x_i),$$

$$w_i^{0,k}(x) = v_i^k \tag{3.11}$$

$$w_i^{\beta,k}(x) = v_i^\beta v_i^k + \frac{1}{2}\sum_{j\neq i}\Psi^{\beta,k}(\varepsilon^{-1}(x_i - x_j)),$$

$$\beta = 1,\ldots,d \tag{3.12}$$

$$w_i^{d+1,k}(x) = v_i^k z_i^{d+1} + \frac{1}{2}\sum_{j\neq i}\sum_{\gamma=1}^{d}\Psi^{\gamma,k}(\varepsilon^{-1}(x_i - x_j))$$

$$\times \frac{1}{2}[v_i^\gamma + v_j^\gamma]\}. \tag{3.13}$$

Here the $\vartheta^{\varepsilon}_{\mu,k}$, with $\mu = 0, \ldots, d+1$ and $k = 1, \ldots, d$ are error terms satisfying the conditions:

$$\vartheta^{\varepsilon}_{0,k} = 0$$

and, for $\mu > 0$,

$$\int_{\mathbb{T}^d_1} dx \sum_{k=1}^{d} \vartheta^{\varepsilon}_{\mu,k}(x) \nabla_k f(x) = O(\varepsilon^2)$$

for any smooth function f.

As in the case of the Boltzmann equation, the currents w cannot be expressed back in terms of the hydrodynamical fields z, and again one has to solve the closure problem in a suitable way to get the hydrodynamical equations. The solution in the Boltzmann case was to use the assumption of local equilibrium which was there justified in terms of scale separation, by the Hilbert expansion and the Caflisch theorem. Hamiltonian particle systems present more difficulties, because the notion of local equilibrium is more delicate in this case and there is no analog of the Caflisch result. We will see that the first difficulty can be solved, while for the second we need to make extra assumptions we cannot check with the present mathematical knowledge or alternatively we need to introduce a little bit of stochasticity in the model [36].

The equilibrium states for particle systems are not just Maxwellians, but Gibbs states.

We fix a small value of ε. A Gibbs state in a macroscopic volume Λ (hence a volume $\varepsilon^{-1}\Lambda$ in microscopic variables) is defined as a probability distribution on the Γ_Λ-space $\bigcup_{N \geq 0} (\Lambda \times \mathbb{R}^d)^N$, whose density with respect to the a priori measure $\frac{1}{n!} d^d x_1 d^d v_1 \ldots d^d x_N d^d v_N$ is given by

$$\begin{aligned} & G_\Lambda(x_1, v_1, \ldots, x_N, v_N) = \\ & \frac{1}{Z_\Lambda} \exp N \int_\Lambda dx \sum_{\alpha=0}^{d+1} \lambda^\alpha z^\alpha(x) \\ & = Z_\Lambda^{-1} \prod_{j=1}^{N} \tilde{z}^N \exp \Big[- \frac{(v_j - u)^2}{2T} \qquad (3.14) \\ & + \frac{1}{T} \sum_{k \neq j} V\Big(\frac{|x_j - x_k|}{\varepsilon} \Big) \Big]. \end{aligned}$$

Here $\tilde{z}$ is the activity, Z_Λ is a normalization factor, $\lambda^\alpha, \alpha = 0, \dots, d+1$ are $d+2$ real numbers called *chemical potentials* associated to the mass, momentum and energy empirical fields and parametrizing the family of the Gibbs states. A more conventional parametrization is given by $\rho > 0$, the *mass density*, $u \in \mathbb{R}^d$, the *velocity field* and $T > 0$, the *temperature*. We denote the set (ρ, u, T) by ζ. The second inequality defines the relation between the two parametrizations, but it is not in general one-valued, because of possible phase transitions. We will restrict our discussion only to the regions of the parameters where the relation between λ and ζ is one to one and smooth and we call the sets λ and ζ conjugate to each other.

The notion of local equilibrium is obtained formally by replacing the constant chemical potentials $\{\lambda^\alpha, \alpha = 0, \dots, d+1\}$ by some smooth functions $\{\lambda^\alpha(x,t), \alpha = 0, \dots, d+1\}$: hence the local equilibrium state is defined, for a fixed ε as

$$G_\varepsilon^{\lambda(\,\cdot\,,t)} = Z^{-1}_{\lambda(\,\cdot\,,t)} \exp \left[N \int_{\mathbb{T}_1^d} dx \sum_{\alpha=0}^{d+1} \lambda^\alpha(x,t) z^\alpha(x) \right], \tag{3.15}$$

with $Z_{\lambda(\,\cdot\,,t)}$ a suitable normalization factor. More explicitly, the restriction of $G_\varepsilon^{\lambda(\,\cdot\,,t)}$ to the N-particle space is given by

$$\begin{aligned} Z^{-1}_{\lambda(\,\cdot\,,t)} \prod_{j=1}^N \tilde{z}^N \exp \Big[& - \frac{(v_j - u(x_j,t))^2}{2T(x_j,t)} \\ & + \frac{1}{T(x_j,t)} \sum_{k \neq j} V\Big(\frac{|x_j - x_k|}{\varepsilon}\Big)\Big], \end{aligned} \tag{3.16}$$

with $\zeta(x,t)$ the conjugate of $\lambda(x,t)$. The procedure adopted here is the same used for constructing the local Maxwellians, but the more complex structure of the Gibbs states requires some comments to convince ourselves that above definition is reasonable.

Roughly speaking, we partition the macroscopic spatial domain $\mathbb{T}_1^d$ with small cubic cells Δ of size $\delta > 0$ centered in points x_Δ. The idea of local equilibrium amounts to assume that in each cell Δ and at any time t the system is distributed according to the Gibbs equilibrium measure with the parameters

λ fixed as $\lambda = \lambda(x_\Delta, t)$. This makes sense provided that the size of the cells is very small compared to the macroscopic length scale, but very large in microscopic unity. More precisely the macroscopic size of the cells is $\varepsilon^{-1}\delta$ and we want it eventually going to infinity. Note that at this qualitative level we don't need to specify more, but when estimates are involved some specific prescription has to be fixed about the size of the cells, typically by taking the limit $\varepsilon \to 0$ with δ fixed first and then the limit $\delta \to 0$. Finally we note that above discussion would justify (3.16) with $\zeta(x_\Delta, t)$ rather than $\zeta(x_j, t)$, but the replacement produces a small error if $x_j \in \Delta$, provided that the size of Δ is macroscopically small. There are several points that would be discussed to make a really convincing definition of a local Gibbs state, but we do not enter in such a discussion, referring rather the reader to [10] for a more detailed discussion. We only note that the most convincing motivation for the above definition would be given by proving that this is a good approximation to the distribution of the system, to the lowest order in ε. This is not known for deterministic Hamiltonian systems, but is true with the addition of some stochasticity [36] and we consider this as a very strong argument in favor of the local equilibrium assumption.

In conclusion, let $F^{(0)}$ denote the density (with respect to the Lebesgue measure) of the initial probability distribution of the system on the space $\Gamma_{\mathbb{T}_1^d}$ and $F^\varepsilon(t)$ its evolution according to the rescaled Liouville equation. We formulate the *local equilibrium assumption* as follows: *The non equilibrium probability distribution F^ε is "well approximated" by a local Gibbs state with smooth parameters as $\varepsilon \to 0$.* We do not specify the meaning of well approximated, but when the assumption is made precise by the proof of a theorem, like in [36], it is stated in terms of the relative entropy that we will discuss in a successive section.

Assuming the local equilibrium and some law of large numbers, is enough to show that the generalized functions z^α and $w^{\alpha,k}$ converge, as ε goes to zero, to their averages, which can be computed using the local Gibbs state and hence can be expressed in terms of the fields ρ, u and T. This provides the closure of the conservation laws (3.3) which become, in the limit $\varepsilon \to 0$ the Euler equations (2.39), with the equations of state for the pressure and the energy,

$$e = e(\rho, T), \quad P = P(\rho, T), \tag{3.17}$$

corresponding to an interacting gas rather than to the ideal gas.

There is no proof of the validity of the Euler equations for deterministic systems (with the remarkable, although rather singular, exception of the one dimensional hard rods model [3]). In [36] it is shown that, if one can prove the decay of the velocity correlations (*local ergodic theorem*), then the local equilibrium assumption becomes correct as $\varepsilon \to 0$ and it is possible to get the Euler equations along the lines explained above, in a region of the parameters where there is no phase transition. For deterministic systems there is not enough control of the correlations to prove the local ergodic theorem, but in [36] the authors add to Newton's equations a stochastic noise, small enough not to modify the macroscopic equations, but sufficient to provide the local ergodicity that they need to complete the proof. The paper has a technical limitation in the fact that the method used is not well suited to deal with high velocities and the result is confined to Hamiltonians growing linearly at high velocities, an example being the relativistic Hamiltonian (see also [34]). As a final remark, let us note that the introduction of some stochasticity is somehow natural because it is commonly accepted that the deterministic dynamics of a many components system presents big instabilities and "sensitive dependence" on the initial conditions, so that the motions of the system appear erratic on the time scales involved and it is impossible to distinguish them from the ones of a stochastic system. Showing that this is really the case is not doable at the present time, but it is reasonable to assume that the effect of the instabilities can be included in the model by the addition of enough stochasticity to destroy the velocity correlations.

In the case of the Navier-Stokes equations we follow the approach which has worked so well in the Boltzmann case, although we do not expect that it can be made rigorous, even under some extra assumptions. By the considerations of Section 2, we confine our presentation to the incompressible case where the scaling provides a criterion to decide which terms have to be retained and which ones are negligible.

In order to enhance the diffusive effects, as we did in Section 2 we look at longer times $\tau = \varepsilon^{-2}t$. We modify previous notation by putting $x_i^\varepsilon(t) = \varepsilon\xi_i(\varepsilon^{-2}t)$ and $v_i^\varepsilon(t) = v_i(\varepsilon^{-2}t)$. They solve the Newton equations

$$\begin{aligned} \frac{dx_i}{dt}(t) &= \varepsilon^{-1}v_i(t) \\ \frac{dv_i}{dt}(t) &= -\varepsilon^{-2}\sum_{i\neq j}\nabla V(\varepsilon^{-1}(x_i - x_j)) \end{aligned} \tag{3.18}$$

for $i = 1, \dots, N$.

The low Mach number assumption consists in choosing as initial distribution $F^{(0)}$ a local Gibbs state which is a perturbation of order ε of a global Gibbs state. This means that its parameters $\zeta_\varepsilon^{(0)}$ are of the form

$$\zeta_\varepsilon^{(0)}(x) = \overline{\zeta} + \varepsilon\zeta_1^{(0)}(x) \tag{3.19}$$

with

$$\overline{\zeta}^\alpha = \begin{cases} 1 & \alpha = 0, \\ 0, & \alpha = k, \quad k = 1, \dots, d, \\ 1 & \alpha = d+1 \end{cases}$$

and

$$\zeta_1^{(0)} = (\rho_0(x), u_0(x), \theta_0(x)). \tag{3.20}$$

Hence the reference temperature and density are set equal to unity without loss of generality, while the mean velocity is assumed initially of order ε by an appropriate choice of the reference frame. Let $\lambda_\varepsilon^{(0)}(x)$ be the conjugate parameters which can be expanded as

$$\lambda_\varepsilon^{(0)}(x) = \overline{\lambda} + \varepsilon\lambda_1^{(0)}(x) + O(\varepsilon^2).$$

Given

$$\lambda_\varepsilon(x,t) = \overline{\lambda} + \varepsilon\lambda_1(x,t) + O(\varepsilon^2)$$

with $\lambda_1(x,t)$ such that $\lambda_1(x,0) = \lambda_1^{(0)}(x)$, it is tempting to consider the local Gibbs state G_t^ε corresponding to such a choice of the chemical potentials and look at the conditions they have to satisfy in order to approximate the true non equilibrium distribution.

As we know from the Boltzmann equation, the local equilibrium assumption is not sufficient to deal with the diffusive case because the transport coefficients involve the corrections to the local equilibrium. Below we present arguments that allow to compute such a correction. Later on we will see that the rigorous counterpart of such arguments, for stochastic lattice systems, is the Varadhan's *non gradient method* [43].

The formal expression for the correction to the local equilibrium is achieved by an expansion technique inspired by the one in Section 2. To do this, it is useful to write the following expression for G_t^ε:

$$G_t^\varepsilon = G_0\Big[1+\varepsilon\big(g_1 - < g_1 > \big) + \varepsilon^2 h_t^\varepsilon\Big], \qquad (3.21)$$

with G_0 the global Gibbs state with chemical potentials $\overline{\lambda}$,

$$g_1 = \int_{\mathbb{T}_1^d} dx \sum_{\alpha=0}^{d+1} \lambda_1^\alpha(x,t) z^\alpha(x) = (\lambda, z),$$

where we use the notation

$$(\phi,\psi) = \int_{\mathbb{T}_1^d} dx \sum_{\alpha=0}^{d+1} \phi_\alpha(x)\psi_\alpha(x),$$

$$< \Phi > = Z_{\overline{\zeta}}^{-1} \sum_{N\geq 0} \int_{\Gamma_N} dx_1 dv_1 \dots dx_N dv_N$$
$$\times\, \Phi(x_1,\dots,v_N)\exp[N\ (\overline{\zeta}, z)]$$

denotes the average of the observable Φ with respect to G_0 and finally h_t^ε is a suitable function of the z^α whose explicit form we do not need.

A key point in the argument is the remark that the empirical fields $z^\alpha(x)$ are *approximate integrals of the motion* in the following sense: we introduce the Liouville operator

$$\mathcal{L}\phi(x,v) = \sum_{i=1}^{N}\sum_{k=1}^{d}\{\varepsilon v_i^k \frac{\partial \phi}{\partial x_i^k} - \sum_{i\neq j} \nabla_k V\Big(\frac{|x_i - x_j|}{\varepsilon}\Big)\frac{\partial\phi}{\partial v_i^k}\} \qquad (3.22)$$

In terms of it, the evolution of any observable Φ according to the Newton equations is given by

$$\partial_t \Phi = \varepsilon^{-2} \mathcal{L} \Phi. \tag{3.23}$$

Clearly $\Phi = \int_{\mathbb{T}_1^d} z^\alpha(x) dx$ are integrals of motion and hence $\mathcal{L}\Phi = 0$. On the other hand, for any smooth function $f(x)$ on $\mathbb{T}_1^d$, by previous calculation it results

$$\mathcal{L}\Big[\int_{\mathbb{T}_1^d} dx f(x) z^\alpha(x)\Big] = O(\varepsilon), \quad \alpha = 0, \ldots, d+1. \tag{3.24}$$

Therefore, while a typical observable has a time derivative of order ε^{-2} the empirical fields associated to the mass, momentum and energy have a time derivative only of order ε^{-1}. So they have a comparatively slow evolution and this justifies the denomination of approximate integrals of motion we used above. We need to assume that the only observables with this property are the empirical fields (and of course functions of them as well). Note that the local Gibbs states corresponding to any choice of smooth chemical potentials, being given by the exponential of linear combinations of the approximate integrals of motion, satisfy the same condition, so that it results

$$\mathcal{L}^\star G_\varepsilon^{\lambda(\,\cdot\,,t)} = O(\varepsilon), \tag{3.25}$$

with $\mathcal{L}^\star = -\mathcal{L}$ the adjoint of $\mathcal{L}$ with respect to the scalar product induced by the Liouville measure. In particular, (3.25) is satisfied by G_t^ε. More specifically, the assumptions on the chemical potential imply that

$$\varepsilon^{-2} \mathcal{L}^\star G_t^\varepsilon = -(\nabla \lambda_1, w) + O(\varepsilon), \tag{3.26}$$

with

$$(\nabla \lambda_1, w) := \int_{\mathbb{T}_1^d} dx \sum_{\alpha=0}^{d+1} \sum_{k=1}^{d} \nabla_k \lambda_1^\alpha(x,t) w^{\alpha,k}(x)$$

After this preparation we try to approximate the non equilibrium distribution F_t^ε which satisfies the Liouville equation

$$\partial_t F_t^\varepsilon = \varepsilon^{-2} \mathcal{L}^\star F_t^\varepsilon. \tag{3.27}$$

We use an approach similar to the Hilbert expansion for the Boltzmann equation: we write F^ε as a series in ε,

$$F^\varepsilon = \sum_{n\geq 0} \varepsilon^n F^n$$

and try to determine the terms of the expansion by substituting it in (3.27). The diverging terms of order ε^{-2} and ε^{-1} are canceled by choosing F_0 as the global equilibrium and F_1 a combination of approximate invariants. To our purposes the only higher order term we need is F_2. Therefore, to simplify the discussion we look at the following form of F_t^ε:

$$F_t^\varepsilon = G_t^\varepsilon + \varepsilon^2 G_0 R_t^\varepsilon \tag{3.28}$$

In fact, by (3.21), G_t^ε also contains second and higher orders in ε, but the distribution of them among G_t^ε and R_t^ε is somewhat arbitrary and the expression(3.28) is convenient for the sequel. Moreover, we can assume that R_ε does not contain combinations of the approximate invariants z^α, because otherwise, such terms can be included in the function h_t^ε.

By substituting (3.3) in (3.27) and taking into account (3.26) and the fact that $\mathcal{L}h_t^\varepsilon = O(\varepsilon)$, because it is a function of the invariants, we conclude that R_t^ε has to solve the equation

$$\mathcal{L}^\star R_t^\varepsilon = (\nabla\lambda_1, w). \tag{3.29}$$

We have some compatibility conditions in order to solve (3.29). First, since $< \mathcal{L}^\star\Phi >= 0$ for any Φ, we need $\langle(\nabla\lambda_1, w)\rangle = 0$. Moreover, if we denote by $Z^\alpha = (1, z^\alpha)$, $\alpha = 0, \dots, d+1$, the global invariants (total mass, total momentum, total energy), we have that the averages w.r.t. G_0 satisfy

$$\langle Z^\alpha \mathcal{L}^\star \Phi\rangle = \langle \mathcal{L} Z^\alpha \Phi\rangle = 0 \tag{3.30}$$

for any smooth observable Φ and for $\alpha = 0, \dots, d{+}1$. Therefore λ_1 has to satisfy the conditions

$$\langle Z^\alpha (\nabla\lambda_1, w)\rangle = 0, \quad \alpha = 0, \dots, d+1. \tag{3.31}$$

Since the average is with respect to the global equilibrium G_0, one can readily check that conditions (3.31) are equivalent to

$$\mathrm{div} u = 0, \quad \nabla(\rho + \theta) = 0 \tag{3.32}$$

as in the Boltzmann case.

Given that such conditions are satisfied, the solution of (3.29) is still of tremendous difficulty in the present framework. We have no hint about the solution, but to complete the argument we assume that such a solution does exist and we denote it by

$$\mathcal{L}^{\star -1}(\nabla\lambda_1, w). \tag{3.33}$$

pretending with it that we can really construct the inverse of the operator $\mathcal{L}^\star$.

We would like to reed the compatibility condition (3.31) as a condition of orthogonality of $(\nabla\lambda_1, w)$ to the space of the invariants, as in the Boltzmann case. To guess what is in this context the right notion of scalar product let us rewrite (3.30) for an observable $\Phi(x)$ with mean zero in the form

$$\int_\Lambda dx\, \langle \tau_x z^\alpha(0)\Phi(x)\rangle = 0,$$

τ_x denoting the translation operator:

$$(\tau_x f)(y) := f(x+y).$$

This suggests to consider the Hilbert space of the local observables equipped with the scalar product [39]

$$(\phi,\psi)_0 = \int dx[\langle \phi\tau_x\psi\rangle - \langle\phi\rangle\langle\psi\rangle] \tag{3.34}$$

By using this notion we can now define the projector on the invariant space as

$$\mathcal{P}\phi = \sum_{\mu=0}^{d+1} (\phi, \hat{z}^\mu)_0 \hat{z}^\mu \tag{3.35}$$

where $\hat{z}^\mu$ is obtained by orthonormalizing the empirical fields $z^\alpha, \alpha = 0, \dots, d+1$, with respect this scalar product. The condition (3.31) is equivalent to

$$\mathcal{P}(\nabla\lambda_1, w) = 0$$

and the equation (3.29) can be interpreted as an equation in the space orthogonal to the invariants, once (3.31) is satisfied.

We can now conclude the formal derivation of the incompressible Navier-Stokes equation. Assuming that F_t^ε is of the form (3.28) with R_t^ε of the form (3.33) we take the average of the momentum current with respect F_t^ε. We get for the currents w's

$$\begin{aligned}
&\left\langle w^{\ell k}(x)\right\rangle_{G_t^\varepsilon}+\varepsilon^2\left\langle w^{\ell k}(x)R_t^\varepsilon\right\rangle\\
=&\left\langle \tilde{w}^{\ell k}(x)\right\rangle_{G_t^\varepsilon}+\varepsilon^2\rho u^\ell u^k+\varepsilon^2\left\langle w^{\ell k}R_t^\varepsilon\right\rangle\\
&=P_0+\varepsilon P_1+\varepsilon^2\left[P_2+\rho u^\ell u^k+\left\langle w^{\ell k}(x)R_t^\varepsilon\right\rangle\right]\\
&+O(\varepsilon^3)
\end{aligned} \tag{3.36}$$

where $< \Phi >_{G_t^\varepsilon}$ is the average with respect to the Gibbs state G_t^ε and $\tilde{w}^{\ell k}$ are given by the expression (3.11), (3.12), (3.13) after replacing the velocities v_i by $\tilde{v}_i = v_i - \varepsilon u(x_i)$.

By the virial theorem [39], the first term in the second line is the thermodynamic pressure in the "equilibrium" state G_t^ε, that we denote by P_ε. The first two terms of the third line of (3.36) do not contribute to INS because P_0 and P_1 are constant (the second equation in (3.32) says that $\nabla P_1 = 0$), while P_2 represents the second order correction to the thermodynamic pressure P_ε and gives rise to the unknown pressure p appearing in the INS.

The fourth term in the last line of (3.36) gives the non linear transport term. Finally, the fifth term is determined by R_t^ε, the non-equilibrium part of the distribution F_ε, and provides the viscous term. Notice that the part of the current $w^{\ell k}$ which is in the invariant space, namely $\mathcal{P}w^{\ell k}$ does not contribute to $\langle w^{\ell k}(x)R_t^\varepsilon\rangle$ since R_t^ε has no component in the invariant space: $\mathcal{P}R_t^\varepsilon = 0$.

Applying the same procedure also to the energy conservation law we get in the limit $\varepsilon \to 0$ the INS equations and the heat equation as given by (2.74), with the factor $5/2$ in the third of eq. (2.74) replaced by c_p the *specific heath at constant pressure* of the fluid and the transport coefficients η and k given by the Green-Kubo formulas. We refer to [17] for details on this.

As mentioned before, to give a rigorous sense to equation (3.29) is a highly non trivial task which has been accomplished in some cases with the help of the Varadhan's non-gradient method [43]. At the moment no effective application of this method to problems where the space variables are continuous is known.

For this reason we try to modify the deterministic model whose evolution is ruled by the Newton's equations not by just adding some stochastic noise, but rather considering a particle system on a lattice. This means that particles are allowed to occupy only lattice sites and move from one site to another via random jumps. The most famous of such models is the *simple exclusion process* which we describe briefly in next section because it is the basis of the construction of the model for the Navier-Stokes equations. We refer the reader to [39] and [26] for a more extensive discussion and references.

4. STOCHASTIC LATTICE GASES.

Lattice gases are particle systems with positions restricted on a lattice; the particles move only by random jumps. We will discuss in this section stochastic lattice gases with hard core exclusion, which means that the number of particles in each site cannot be arbitrary large. The simplest and most famous is the so called simple exclusion process, where at most one particle per site is allowed. In the sites of a d-dimensional lattice $\mathbb{Z}^d$ there are clocks ringing at times exponentially distributed, independently each others. When the clock in a site x occupied by a particle rings, the particle jumps to one of the nearest sites y at some given rate $p(x,y) \geq 0$ if it is empty; otherwise stays in x and waits for the next ring. We denote by $\eta(x) = 0, 1$ the occupation number at site x and by $\eta \in \Omega_\Lambda = \{0,1\}^\Lambda$ a configuration of the system on a subset Λ of the lattice. We will use $\Omega \equiv \Omega_{\mathbb{Z}^d}$. If η is the configuration before the jump then after the jump from x to y the new configuration is

$$\eta^{x,y}(z) = \begin{cases} \eta(z) \text{ if } z \neq x, y \\ \eta(x \text{ if } z = y \\ \eta(y) \text{ if and } z = x \end{cases}$$

The jump rate given by

$$c(x, y, \eta) = p(x, y)\eta(x)(1 - \eta(y)) \qquad (4.1)$$

ensures the hard core condition. The generator for this stochastic dynamics is defined as follows: for any local function f (i.e. for any function depending only on the configuration η in a finite set) we put

$$\mathcal{L}f(\eta) = \sum_{x,y\in\mathbb{Z}^d} c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)] \qquad (4.2).$$

The total number of particles is conserved during the motion and this is the only conservation law for this dynamics. Invariant measures for the dynamics are the product measures μ^λ whose marginals in Ω_Λ have density (with respect to the counting measure) given by

$$G_\Lambda^\lambda(\eta) = \prod_{x\in\Lambda} \frac{e^{\lambda\eta(x)}}{1+e^\lambda}. \qquad (4.3)$$

This family of invariant measure is parametrized by the chemical potential $\lambda \in \mathbf{R}$. The Markov process corresponding to the generator (4.2) is reversible with respect these invariant measures if the jump rate verifies $p(x,y) = p(y,x)$. In this case the process is called *symmetric simple exclusion.* Otherwise, the process is called *asymmetric simple exclusion* and it is not reversible.

We introduce the current through the bond (x,y) as the rate at which a particle jumps from x to y minus the rate at which a particle jumps from x to y

$$w_{x,y} = [c(x,y,\eta) - c(y,x,\eta)] \qquad (4.4)$$

If the process is symmetric the current has a nice expression

$$\begin{aligned} w_{x,y} &= p_{x,y}[\eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x))] \\ &= p_{x,y}[\eta(x) - \eta(y)] \end{aligned} \qquad (4.5)$$

namely it can be thought of as the lattice gradient of $-\eta(x)$. Models of particles on the lattice having the property that the current is the gradient of some function are called *gradient models* [39]. For this class of models it is by now relatively simple to get the hydrodynamical equation on the diffusive time scale. Let us explore this important fact for the exclusion process. Calling ε as usual the scale separation parameter we speed up the time as ε^{-2}.

The empirical density field on the lattice is defined as

$$\nu^\varepsilon(z,t) = \varepsilon^d \sum_{x\in\Lambda} \delta(z-\varepsilon x)\eta(x,\varepsilon^{-2}t)$$

where z and t are the macroscopic coordinate and time, $\Lambda \in \mathbb{Z}^d$ is a cubic sublattice $\{-L,\dots,L\}^d$ and $L=\varepsilon^{-1}$ with periodic boundary conditions. ν^ε is expected to approximate a smooth density profile $\rho(z,t)$ for ε small. Consider a smooth test function J on the unit torus $\mathbb{T}_1^d$, so that $J(\varepsilon x)$ is a slowly varying function on the microscopic scale. We have, by definition,

$$\int_\Lambda J(z)\nu^\varepsilon(z,t)dz = \varepsilon^d \sum_{x\in\Lambda} J(\varepsilon x)\eta(x,\varepsilon^{-2}t)$$

The measure ν^ε satisfies a local conservation law expressed by

$$\begin{aligned}&\frac{d}{dt}\varepsilon^d \sum_{x\in\Lambda} J(\varepsilon x)\eta(x,\varepsilon^{-2}t) = \\ &-\varepsilon^{d-2}\sum_{x\in\Lambda} J(\varepsilon x)\sum_{e>0}\nabla_e^- w_{x,x+e} + M_t\end{aligned} \tag{4.6}$$

where e denote a unit vector on the lattice, $\sum_{e>0}$ means the sum is restricted to the unit vectors with non negative components and

$$\nabla_e^- g(x) := g(x)-g(x-e),\ \nabla_e g(x) := g(x+e)-g(x).$$

Moreover, M_t is a stochastic noise, technically a martingale, and hence $\mathbb{E}_t[M_s]=0$ for any $s>t$, $\mathbb{E}_t$ being the conditioned expectation to configuration at all the times $t'\le t$.

After "integrating by part" on the lattice, by the periodic boundary conditions on Λ we get

$$\begin{aligned}&\frac{d}{dt}\varepsilon^d \sum_{x\in\Lambda} J(\varepsilon x)\eta(x,\varepsilon^{-2}t) = \\ &\varepsilon^{d-2}\sum_{x\in\Lambda}\sum_{e>0}\nabla_e J(\varepsilon x) w_{x,x+e} + M_t = \\ &-\sum_{x\in\Lambda}\sum_{e>0}\nabla_e J(\varepsilon x)\nabla_e \eta(x,\varepsilon^{-2}t) + M_t.\end{aligned}$$

By using (4.5) we can integrate again by part and get

$$\frac{d}{dt}\varepsilon^d \sum_{x\in\Lambda} J(\varepsilon x)\eta(x,\varepsilon^{-2}t) =$$
$$\varepsilon^{d-2}\sum_{x\in\Lambda}\frac{1}{2}\sum_{e>0} p_e \nabla_e^- \nabla_e J(\varepsilon x)\eta(x,\varepsilon^{-2}t) + M_t. \tag{4.7}$$

We can now approximate the lattice gradient of J by the gradients with respect the macroscopic variable getting a factor ε^2 compensating the divergent factor ε^{-2}. To get the macroscopic equation for the density we take the non equilibrium expectation of both sides of (4.7). Fixed once and for ever a value of λ, we denote the corresponding product measure (4.3) by μ and consider initial measures absolutely continuous with respect to μ. The non equilibrium probability measure at time t has a density f_t with respect to μ satisfying the Fokker-Plank equation

$$\frac{\partial f_t}{\partial t} = \varepsilon^{-2}\mathcal{L}^\star f_t,$$

$\mathcal{L}^\star$ being the adjoint of the generator with respect to the invariant measure (note that $\mathcal{L}^\star = \mathcal{L}$ in this case because of the reversibility).

We now introduce the *local equilibrium assumption*: fix a smooth function $\lambda(z,t)$ on $\mathbb{T}_1^d \times \mathbf{R}_+$, and consider the expression in the r.h.s. of (4.3) with λ replaced by $\lambda(\varepsilon x, t)$ as we did in previous section: this means that the density of $\mu^{\lambda(\,\cdot\,,t)}$ is given by

$$G_\Lambda^{\lambda(\,\cdot\,,t)}(\eta) = \prod_{x\in\Lambda}\frac{\mathrm{e}^{\lambda(\varepsilon x,t)\eta(x)}}{1+\mathrm{e}^{\lambda(\varepsilon x,t)}}. \tag{4.8}$$

We call $\mu^{\lambda(\,\cdot\,,t)}$ the *local equilibrium* corresponding to $\lambda(z,t)$. We assume that for ε small f_t is close to $\mu^{\lambda(\,\cdot\,,t)}$.

By noting that $\mathbb{E}^{\mu^{\lambda(\,\cdot\,,t)}}[\eta(x)] = \phi(\lambda(\varepsilon x,t)) := \rho(\varepsilon x,t)$, where $\phi(\lambda) = \mathrm{e}^\lambda/(1+\mathrm{e}^\lambda)$ is the density conjugate to λ, in the limit $\varepsilon \to 0$ we get

$$\frac{d}{dt}\int dz J(z)\rho(z,t) = \int dz \sum_{i=1}^d p_i \frac{\partial^2}{\partial z_i^2} J(z)\rho(z,t),$$

with $p_i = p(x, x+e-i)$, e_i the unit vector in the i-th coordinate direction. This is the weak form of the diffusion equation for the density

$$\frac{\partial}{\partial t}\rho(z,t) = \sum_{i=1}^{d} p_i \frac{\partial^2}{\partial z_i^2}\rho(z,t).$$

Above conclusion has been made rigorous in several paper and we refer to [39], [26], [11] for references.

In the asymmetric case the current has the expression

$$\begin{aligned} w_{x,y} &= \eta(x)\eta(y)[p(x,y) - p(y,x)] \\ &\quad - \eta(x)p(x,y) + \eta(y)p(y,x). \end{aligned}$$

Define the symmetric and antisymmetric part of p as

$$\begin{aligned} p^s(x,y) &= \frac{p(x,y)+p(x,y)}{2}, \\ p^a(x,y) &= \frac{p(x,y)-p(x,y)}{2}. \end{aligned}$$

Then

$$\begin{aligned} w_{x,y} &= p^s(x,y)[\eta(y)-\eta(x)] \\ &\quad + p^a(x,y)[2\eta(x)\eta(y) - (\eta(x)+\eta(y)) \qquad (4.9) \\ &:= w^s(x,y) + w^a(x,y), \end{aligned}$$

which we call *symmetric and antisymmetric parts of the current* respectively.

The antisymmetric part of the current is not the gradient of some function, so that the previous argument to get the limiting equation does not work: it is not possible "to integrate twice by part" and the contribution of the antisymmetric part of the current to the local conservation law (4.6) seems to be of order ε^{-1}.

Suppose that it is possible to write the current as

$$w_{x,x+e} = \sum_{e'>0} D_{e,e'}[\eta(x+e') - \eta(x)] + \tau_x \mathcal{L} F_e \quad (4.10)$$

for some suitable coefficients independent of the configuration but possibly depending on the density and some suitable vector function F whose components are F_e. τ_x stands for the spatial shift by x. The contribution of the second term in (4.10) to (4.6) after integration over time on an interval $[0,T]$ is

$$-\varepsilon^{d-2}\int_0^T dt \sum_{x\in\Lambda}\sum_{e>0} p_e \nabla_e J(\varepsilon x)\tau_x \mathcal{L} F_e \qquad (4.11)$$

By taking the expectation with respect f_t we get

$$\varepsilon^{-2}\int_0^T dt \mathbb{E}^{f_t}[\mathcal{L}F_e] = \varepsilon^{-2}\int_0^T dt \mathbb{E}^{\mu}[\mathcal{L}^\star f_t F_e]$$
$$= \mathbb{E}^{f_T}[F_e] - \mathbb{E}^{f_0}[F_e].$$

Hence the term (4.11), after replacing the discrete lattice derivative with continuous derivative, turns out to be of order ε and disappears in the limit.

The other term in the continuous limit gives a contribution of the form

$$\sum_{j=1}^d \frac{\partial}{\partial z_j} D_{i,j}(\rho(z,t)).$$

Equation (4.10) has to be understood as an equation in a suitable Hilbert space and the coefficients $D_{i,j}$ have the geometric interpretation as "the component of the currents in the direction of the gradients of conserved quantities". This Hilbert space represents the space of the observables "orthogonal" to the conserved quantity (the density). It is the analogous of the space of the non hydrodinamical quantities for the linearized Boltzmann operator.

The previous argument, showing that the term $\tau_x \mathcal{L} F_e$ in (4.10) gives a contribution of order ε while the gradient term is of order 1, suggests that the "gradients" are slower modes of the dynamics compared to terms of the form $\mathcal{L}h$. For this reason we will refer to (4.10) as "slow-fast modes decomposition".

It is possible to prove a decomposition (4.10) for some non gradient lattice gases [26] and in particular for the asymmetric part of the current associated to the simple exclusion process (the symmetric part being obviously a gradient). We note that solving the many-body equation (4.10) means finding both the function F_e and a matrix D such that $\mathcal{L}F_e = U_e$, with $U_e = w_{0,e} + \sum_{e'>0} D_{e,e'}[\eta(e') - \eta(0)]$. This problem is analogous to the problem of solving the equation $\mathcal{L}f = g$ in the set-up of the Boltzmann equation to compute the first correction in the Hilbert expansion.

We will explain in next section how to construct the Hilbert space where it is possible to give sense to (4.10). Here, we focus on the problem of getting the hydrodynamic limit for the case of the asymmetric simple exclusion. An extra difficulty is that the current w is not in the appropriate Hibert space. However, by suitable subtractions it is always possible to construct currents $\sigma_{x,x+e} = w_{x,x+e} - g_{x,x+e}$ belonging to the space. The term $g_{x,x+e}$ that we have to subtract to the current for the decomposition (4.10) to work is essentially (in the limit) the expectation with respect to $\mu^{\lambda(\cdot,t)}$ of the current and since in (4.6) the current is multiplied by ε^{-2} it gives rise to a macroscopic current of the form

$$\varepsilon^{-1}\delta[\rho(1-\rho)]$$

where δ is the vector with components $\delta_i = p_{e_i} - p_{-e_i}$, $i = 1, \ldots, d$. This term appears in the macroscopic equation as a transport term of the first order in the spatial derivatives and this is the reason of the factor ε^{-1}. This contribution is in general too big and hence it is not possible to get a limiting equation in the diffusive scaling of the asymmetric simple exclusion. This is just the analog of the diverging Euler modes appearing when trying to get the Navier-Stokes equation from the Boltzmann equation in the compressible case. On time scales of order ε^{-2} the density waves propagate with velocity of order ε^{-1}. If δ is of order ε, which corresponds to the *weakly asymmetric simple exclusion process*, the macroscopic current becomes of order ε and the limiting equation can be proven to be just a diffusion with a (non linear) drift ([12], [27]). In this case, however, the asymmetric nature of the problem disappears and the diffusion coefficient so obtained is the same as the symmetric one. If δ is not of order ε one can try an approach similar to the one used for the incompressible limit of the Boltzmann equation. We choose a suitable initial state for the system so that the density is a perturbation of order ε of a homogeneous profile: suppose that the initial density $\rho_0(z)$ is of the form

$$\rho_0(\varepsilon x) = \theta - \varepsilon u_0(z)$$

for some constant $\theta \in (0,1)$. Then, assuming that this form of the density persists at time t, the macroscopic current would be

$$J = \delta\theta(1-\theta) - \varepsilon v u - \varepsilon^2 \delta u^2,$$

where $v = (1-2\theta)\delta$ is a constant vector parallel to δ. Therefore the equation for the perturbation of the density would become:

$$\partial_t u + \varepsilon^{-1} v \cdot \nabla_z u + \delta \cdot \nabla_z u^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 u}{\partial z_i \partial z_j} \quad (4.12)$$

with $D_{i,j}$ determined by (4.10). A diverging term is left in (4.12), but is a trivial divergence in the sense that a global "Galileian" transformation, corresponding to set

$$m(z,t) = u(z + \varepsilon^{-1} vt, t),$$

removes it and the equation for $m(z,t)$ is the *viscous Burgers equation*

$$\partial_t m + \delta \cdot \nabla_z m^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 m}{\partial z_i \partial z_j} \quad (4.13)$$

Note that in the case $\theta = \frac{1}{2}$ no Galileian transformation is necessary because $v = 0$.

In [18] it is proven that the hydrodynamical limit of the (asymmetric) simple exclusion is indeed the equation (4.13). The precise statement of the result is as follows. Assume that the probability measure at time zero has the form

$$\Psi_0(\eta) = \prod_{x \in \Lambda} \frac{\mathrm{e}^{r + \varepsilon \lambda_0(\varepsilon x)\eta(x)}}{1 + \mathrm{e}^{r + \varepsilon \lambda_0(\varepsilon x)\eta(x)}}$$

where $r := \phi^{-1}(\theta)$. Assume that $m(z,t)$ is a classical solution of (4.13) corresponding to the initial datum

$m_0(z) = \varepsilon^{-1}[\phi(r + \varepsilon\lambda_0(\varepsilon x)) - \theta]$. Let $\nu_\varepsilon(z,t)$ be the empirical measure

$$\nu_\varepsilon(z,t) = \varepsilon^{d-1} \sum_x \delta(z - \varepsilon x)(\theta - \eta(x, \varepsilon^{-2} t)).$$

and $\hat{\nu}_\varepsilon(z,t) = \nu_\varepsilon(z - \varepsilon^{-1} vt, t)$. Next theorem states the weak convergence of $\hat{\nu}_\varepsilon(z,t)$ to $m(z,t)$, if $d \geq 3$.

Theorem 4.1 *For $d \geq 3$, $\hat{\nu}_\varepsilon(z,t)$ converges weakly in probability to $m(z,t)$ solving the equation (4.13). The diffusion matrix satisfies $1 < D < \infty$ (as a matrix) and it is given by the Green-Kubo formula.*

$$D_{i,j} = \delta_{i,j} + (2\chi)^{-1} \int_0^\infty dt \sum_{x \in \mathbb{Z}^d} \mathbb{E}^\mu [\sigma_{e_i} \mathrm{e}^{\mathcal{L}t} \sigma_{x,x+e_j}] \tag{4.14}$$

where μ is the invariant measure with $\lambda = \phi(\theta)$ and $\chi = \theta(1-\theta)$ is the compressibility.

The restriction on the dimensions is optimal since in this scaling the diffusion coefficient is expected to be infinite for $d = 1$ or 2. It was conjectured that the correct scaling for $d = 1$ is $\varepsilon^{-3/2}$ while for $d = 2$ there is a logarithmic correction to ε^{-2}. See [39] for a review and references therein.

It should be emphasized that the diffusion coefficient for the asymmetric process is strictly bigger (as a matrix) than the corresponding one for the symmetric process, namely the process with jump intensities $\tilde{p}_{\pm e} = (p_e + p_{-e})/2$. To understand the meaning of this fact we notice that the antisymmetric part of the simple exclusion process can be interpreted as a deterministic motion, while the symmetric part represents the stochastic noise present in the system. In this respect the extra diffusivity arising in the asymmetric case is determined by a mechanism that is reasonable to compare to the one expected to produce viscosity in deterministic Hamiltonian systems. The relevance of the statement $D > 1$ is that even such a simple model captures this kind of mechanism producing more diffusivity than what is initially embedded in the system.

Before discussing more complex models, let us mention that, while for the symmetric simple exclusion the Euler time scale is trivial, for the asymmetric simple exclusion the Euler scaling is interesting. In fact, because of the drift term, the limiting equation, in the hyperbolic scaling, is non linear and coincides with the inviscid Burgers equation, usually called simply *Burgers equation*. The local equilibrium argument readily provide justification to this statement and a rigorous proof in the smooth regime is non difficult to achieve. However the Burgers equation is known to develop singularity of shock type in finite time. It is remarkable that even in this case it is possible to show that the hydrodynamical limit exists and corresponds to the "entropic solutions" of the Burgers equation [38].

Still on the Euler time scale it is conceivable to look at the first order corrections to the Burgers equation along the same path followed to get the compressible Navier-Stokes equations from the Boltzmann equation. The resulting equation is the Burgers equation with a non linear diffusion term of order ε determined by the diffusion matrix (4.14). This can be achieved with the methods used in [18], but requires a more detailed analysis provided in [30].

Since the diffusive correction in the Euler time scale is of order ε, to get sharp informations on the limiting behavior one would like to extend above analysis to the parabolic time scale. This is related to the question of the Navier-Stokes correction to the Euler equations and its ambiguity [29]. The problem is very difficult by the presence of a macroscopic current of order ε^{-1} and only for small perturbations of the constant density profile the limit has been proven to be meaningful. As mentioned before this is the analog of the incompressible Navier-Stokes setting.

The incompressible approximation for the hydrodynamical equations, which means to consider density, velocity and temperature constant up to terms of order ε and look to the equations for the corrections, has indeed a well defined limit as ε goes to 0. It is actually this feature that makes possible to prove that the incompressible Navier-Stokes equations are the diffusive scaling limit of the Boltzmann equation (see [9], [1]). Since this program turns out to be successful also for the asymmetric simple exclusion process, it is natural to look for particle systems on the lattice more closely modeling the macroscopic behavior of a viscous fluids. Examples of such systems are provided by the so called *cellular automata* [22] and the models we consider are inspired by them, with suitable modifications.

5. NAVIER-STOKES EQUATION.

To construct a lattice gas modeling a real fluid we need a dynamics that conserves not only the particle number but also momentum and energy. For that we need a model of particles that, like real particles, have velocity. For stochastic particles the notion of velocity is replaced by the drift. This is the vector $\sum_{e\in\mathcal{D}} ep(x, x+e)$, where $p(x, x+e) := p_e$ is the jump rate in the direction $e \in \mathcal{D} = \{e \mid e = \pm e_i, i = 1, \dots, d\}$, e_i, $i = 1, \dots, d$ being the unit vectors in the positive coordinate directions. Therefore a particle moving on the lattice with velocity v may be modeled by a simple exclusion process with jump rates $p_e(v)$ such that the corresponding drift equals v:

$$\sum_{e\in\mathcal{D}} ep_e(v) = v.$$

We will include in the model several velocities and the properties of the model depend strongly of what is the choice of the set of velocities. We present the 3-dimensional model studied in [2], which is an extension of the one introduced in [20], suitable to include not only fluid motions but also thermal phenomena.

Let $\Lambda \in \mathbb{Z}^d$ be, as usual, the cubic sublattice $\{-L, \dots, L\}^d$ with periodic boundary conditions. We set $L = \varepsilon^{-1}$. Let $\mathcal{V}$ be a finite subset of $\mathbb{R}^d$ representing the possible velocities and N the numbers of elements in $\mathcal{V}$. Assume that $\mathcal{V}$ is invariant under reflections and permutations of the coordinate axes.

Most of the discussion below does not depend on the specific choice of $\mathcal{V}$ but for the *finite volume ergodic property* we will need below. To fix the ideas one can think to the following choice: $d = 3$ and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $\mathcal{V}_1 = \{(\pm1, \pm1, \pm1)\}$ and $\mathcal{V}_2 = \{(\pm1, \pm1 \pm \omega)\}$, up to permutations, with the constant $\omega \neq 1$ to be chosen in a suitable way to get the Navier-Stokes equations. We have 8 velocities of the first type whose kinetic energy is $3/2$ and 24 of the second one with kinetic energy $(2+\omega^2)/2$. Therefore, in this model there are different kinetic energies and it is possible to simulate thermal effects in the hydrodynamic equations. We will refer to the model in [2] as *thermal lattice gas*. The model considered in [20] on the other hand, contains only velocities in $\mathcal{V}_2$. Particles have fixed kinetic energy and the model is useful to simulate the fluid motion without thermal transfers. We will refer to this model as *model* $\mathcal{V}_2$. Note that the value of ω that provides the Navier-Stokes equations in model $\mathcal{V}_2$ is not the same of the thermal lattice gas.

On each site of the lattice at most one particle for each velocity is allowed, so that there are at most N particles per site (exclusion rule only between particles with the same velocity). A configuration of particles on the lattice is denoted by $\eta = \{\eta_x, x \in \Lambda_L\}$ where $\eta_x = \{\eta(x,v), v \in \mathcal{V}\}$ and $\eta(x,v) = 0,1$ is the occupation number of particles at x with velocity v. The configuration space in Λ is denoted by $\Omega_\Lambda = \{0,1\}^{\Lambda\times\mathcal{V}}$. Moreover, $\Omega = \Omega_{\mathbb{Z}^d}$.

The dynamics consists of the following two parts:

1- *Free motion*: the motion of particles with velocity v is modeled by a simple exclusion process (SEP) with drift v. In other words, particles with velocity v jump at (independent) exponential times from a site x to a neighboring site $x+e$, $e \in \mathcal{D}$, with intensity $p(x, x+e, v)$ provided that there is no particle with velocity v at $x+e$. The intensity is chosen so that the drift $\sum_{e\in\mathcal{D}} ep(0,e,v) = v$. This is the way we associate a notion of velocity to particles on the lattice and one can obtain any velocity v by choosing the jump rate correctly even if the underlying lattice is the standard cubic lattice. This is the main difference between this model and cellular automata considered in [22] where the velocity set and the geometry of the lattice have to be related. The price we pay is that our models are essentially not deterministic, what makes them less useful for numerical simulations, but much more treatable from the theoretical point of view.

We choose $p(x,y,v) = [\gamma + (y-x)\cdot v/2]$, where γ is a constant large enough so that the jump rates are nonnegative. One can characterize this dynamics formally by the generator

$$\mathcal{L}^{ex} f(\eta) = \sum_{\substack{x\in\Lambda,\\ e\in\mathcal{D}, v\in\mathcal{V}}} \eta(x,v)(1-\eta(x+e,v))\times$$
$$p(x,x+e,v)\big[f(\eta^{x,x+e,v})-f(\eta)\big]$$

where $\eta^{x,y,v}$ is the configuration obtained from η by exchanging the occupation numbers of particles with velocity v at x and y.

2- *Collisions*: the velocity of the particles change by means of binary collisions. By collisions we mean all the quadruples

$$\mathcal{Q} = \Big\{q \in \mathcal{V}^4 \ |q = (v,w,v',w'),$$
$$v+w = v'+w', \quad v^2+w^2 = v'^2+w'^2\Big\}$$

conserving the total momentum and the total kinetic energy. Here v,w are the incoming velocities and v',w' the outgoing velocities. These binary collisions occur at a site at independent exponential times, provided the exclusion rule is not violated.

The generator is given by

$$\mathcal{L}^{c} f(\eta) = \sum_{\substack{x\in\Lambda,\\ q\in\mathcal{Q}}} C(x,q,\eta)\big[f(\eta^{x,q})-f(\eta)\big],$$

where $C(x,q,\eta) = 1$ if $\eta(x,v) = \eta(x,w) = 1$ and $\eta(x,v') = \eta(x,w') = 0$, $C(x,q,\eta) = 0$ otherwise. Note that $C(x,q,\eta) = 0$ when in the configuration η there is no particle with velocity v or w at x or there is already a particle with velocity v' or w' at x. The configuration $\eta^{x,q}$ is constructed by setting $\eta^{x,q}(x,v) = \eta^{x,q}(x,w) = 0$, $\eta^{x,q}(y,v') = \eta^{x,q}(y,w') = 1$ and $\eta^{x,q}(z,u) = \eta(z,u)$ if $z \neq x$ or $u \notin q$.

The generator of the full dynamics is defined as

$$\mathcal{L} = \mathcal{L}^{ex} + \mathcal{L}^{c}$$

There are $d+2$ conserved quantities for the dynamics, namely, the total mass, the d components of the total momentum and the total kinetic energy.

Denote by $I_\alpha, \alpha = 0, d+1$ the total mass, the total momentum and the total energy at the site x:

$$I_0(\eta_x) = \sum_{v \in \mathcal{V}} \eta(x,v) \tag{5.1}$$

$$I_\alpha(\eta_x) = \sum_{v \in \mathcal{V}} (v \cdot e_\alpha)\eta(x,v)\,,\ \alpha = 1, \ldots, d \tag{5.2}$$

$$I_{d+1}(\eta_x) = \sum_{v \in \mathcal{V}} \frac{1}{2}|v|^2 \eta(x,v) \tag{5.3}.$$

By *finite volume ergodic property* we mean that for any finite region W of the lattice, the dynamics restricted to this region (meaning that jumps can occur only between sites in W) conserves only the quantities $\sum_{x \in W} I_\alpha(x)$, $\alpha = 0, \ldots, d+1$. In other words, if η and ζ are two configurations in Ω_W with the same values of the invariants, then it is possible to transform η in ζ by performing only jumps and collisions in W. We assume that our model satisfies such a property. It is proved in [20] that this is the case for the model with $\mathcal{V} = \mathcal{V}_2$, while the same is proved in [2] for the thermal lattice gas. In both cases, ω^2 is assumed irrational.

For any $n \in \mathbf{R}^{d+2}$ introduce the equilibrium state μ^n with marginals in Ω_Λ, μ^n_Λ, having densities

$$\mu^n_\Lambda(\eta) = Z^{-1}_{\Lambda,n} \prod_{x \in \Lambda} \exp\Big\{\sum_{\beta=0}^{d+1} n_\beta I_\beta(\eta_x)\Big\} \tag{5.4}$$

where $Z_{L,n}$ is a normalization factor. In particular, we denote by μ^0_Λ the one corresponding to $n = (r, 0, \ldots, 0, \theta)$:

$$\mu^0_\Lambda(\eta) = Z^{-1}_{L,r,\theta} \prod_{x \in \Lambda} \exp\{rI_0(\eta_x)\} + \theta I_{d+1}(\eta_x)\}. \tag{5.5}$$

This measure will be used as reference measure. It is parametrized by the "chemical potentials" (r, θ), $r \in \mathbf{R}$. All the measures (5.4) are absolutely continuous with respect to the measure μ^0_Λ and are parametrized by a $d+2$-dimensional vector n of *chemical potentials*, which are the variables conjugate to the density, velocity and temperature, as in the Hamiltonian case discussed in Section 3. We define the vector $m \in \mathbf{R}^{d+1}$ by

$$m_\alpha = \mathbb{E}^{\mu^n}[I_\alpha(x)], \quad \alpha = 0, \ldots, d+1.$$

Hence m_0 is the expected value of the mass density in the equilibrium state μ^n, m_α the one of the α-th component of the momentum density and m_{d+1} the one of the energy density. Note that the map $n \to m(n)$ is one to one and smooth because the measure μ^n is a product measure. When dealing with non product measures one should restrict to the one-phase region in order to make this true. We also denote by $n(m)$ the inverse of this map.

Like for Hamiltonian systems, our dynamics is not *reversible* with respect to the measure μ^0_Λ, i.e., the *detailed balance condition*, stating that the adjoint $\mathcal{L}^\star$ of $\mathcal{L}$ with respect to the measure μ^0_Λ coincides with $\mathcal{L}$, is not satisfied.

On the othe hand, the relation $\mathcal{L}^\star = -\mathcal{L}$ (which is true for the Liouvlle operator w.r.t the Gibbs measure) is satisfied by the anysimmetric part of the generator $\frac{\mathcal{L}-\mathcal{L}^\star}{2}$. Hence we will think of the symmetric part of $\mathcal{L}$ as the stochastic component of the motion and of the antysimmetric part as the deterministic component. Moreover, this dynamics is invariant under *time reversal* in the sense that

$$\mathcal{L}S = S\mathcal{L}^\star \tag{5.6}$$

where S is the velocity flip operator defined as

$$(Sf)(\eta) = f(S\eta), \quad S\eta = \{\eta(x,-v), x \in \Lambda, v \in \mathcal{V}\}\}.$$

For any $\beta = 0, \dots, d+1$ and $\alpha = 1, \dots, d$, we denote by $w^\beta_{x,\alpha}$ the current of the conserved quantity I_β in the direction e_α at site $x \in \Lambda$. They are local functions such that

$$\mathcal{L}\big[I_\beta(\eta_x)\big] = -\sum_{\alpha=1}^{d} \nabla^-_\alpha w^\beta_{x,\alpha} \tag{5.7}$$

Since the collision operator conserves the quantities $I_\beta(\eta_x)$ for any $x \in \Lambda$, there is no contribution to the currents coming from $\mathcal{L}^{\mathrm{c}}$:

$$\mathcal{L}\big[I_\beta(\eta_x)\big] = \mathcal{L}^{\mathrm{ex}}\big[I_\beta(\eta_x)\big]. \tag{5.8}$$

Similarly we define the currents $w^{*,\beta}_{x,\alpha}$ for the adjoint operator $\mathcal{L}^\star$:

$$\mathcal{L}^\star\big[I_\beta(\eta_x)\big] = -\sum_{\alpha=1}^{d} \nabla^-_\alpha w^{\star,\beta}_{x,\alpha}. \tag{5.9}$$

The currents can be written as a sum of a symmetric part and an antisymmetric part. For $\alpha = 1, \dots, 3$ and $\beta = 0, \dots, d+1$

$$w^{\beta}_{x,\alpha} = -\gamma \nabla_\alpha I_\beta(\eta_x) + w^{(a),\beta}_{x,\alpha} \tag{5.10}$$

the antisymmetric mass current is

$$w^{(a),0}_{x,\alpha} = \sum_{v \in V} w^{(a)}_{x,\alpha}(v) = \sum_{v \in \mathcal{V}} (e_\alpha \cdot v) b_{x,\alpha}(v), \tag{5.11}$$

where $b_{x,\alpha}(v)$ is the antisymmetric part of the current in the direction e_α for the simple exclusion process with drift v:

$$\begin{aligned} b_{x,\alpha}(v) = &-\eta(x,v)\eta(x+e_\alpha, v) \\ &+ \frac{1}{2}(\eta(x,v) + \eta(x+e_\alpha, v)). \end{aligned}$$

The antisymmetric part of the momentum current is, for $\alpha, \beta = 1, \dots, d$

$$\begin{aligned} w^{(a),\beta}_{x,\alpha} &= \sum_{v \in V} (e_\beta \cdot v) w^{(a)}_{x,\alpha}(v) \\ &= \sum_{v \in \mathcal{V}} (e_\alpha \cdot v)(e_\beta \cdot v) b_{x,\alpha}(v), \end{aligned} \tag{5.12}$$

and the antisymmetric part of the energy current is, for $\alpha = 1, \dots, d$

$$\begin{aligned} w^{(a),d+1}_{x,\alpha} &= \sum_{v \in V} \frac{1}{2} |v|^2 w^{(a)}_{x,\alpha}(v) \\ &= \sum_{v \in \mathcal{V}} \frac{1}{2} (e_\alpha \cdot v) |v|^2 b_{x,\alpha}(v) \end{aligned} \tag{5.13}$$

In the same way the currents associated to the adjoint operator $\mathcal{L}^*$ are given by

$$w^{*,\beta}_{x,\alpha} = -\chi \nabla_\alpha I_b(\eta_x) - w^{(a),\beta}_{x,\alpha} \tag{5.14}$$

with $\alpha = 1, \dots, 3$ and $\beta = 0, \dots, 4$.

Now we define the space where the analog of equation (4.10) for this model has to be solved. Fixed

$m \in \mathbf{R}^{d+2}$, set $\mu_m = \mu^{n(m)}$. Define the space $\mathcal{G}$ as the space of the local functions on Ω satisfying

$$I\!\!E^{\mu_m}[g] = 0, \quad \frac{\partial I\!\!E^{\mu_m}[g]}{\partial m_\alpha} = 0, \quad \alpha = 0, \dots, d+1. \tag{5.15}$$

This condition implies that the elements of $\mathcal{G}$ have no component along the null space of the generator $\mathcal{L}$, the space of the "hydrodynamical variables".

We denote by $\sigma^\beta_{x,\alpha}$ the part of the current $w^\beta_{x,\alpha}$ in $\mathcal{G}$. Therefore we have the following decomposition of the currents into "non hydrodynamical" and "hydrodynamical" parts:

$$w^\beta_\alpha = \sigma^\beta_\alpha + g^\beta_\alpha,$$

where we omit the label x because, by translation invariance we can fix $x = 0$. The terms g^β_α will give rise to the non linear terms in the hydrodynamical equations. We will see in next section that they can be computed as the expectations of the currents in an equilibrium state of the form (5.4) in infinite volume.

We will also state a theorem on the validity in a Hilbert space which is a suitable completion of the space $\mathcal{G}$ of the slow-fast modes decomposition

$$\sigma^\beta_\alpha = \sum_{\gamma=0}^{d+1} \sum_{\vartheta=1}^{d} D^{\beta,\gamma}_{\alpha,\vartheta} \nabla_\vartheta I_\gamma + \mathcal{L} F^\beta_\alpha \tag{5.16}$$

As in the case of ASEP the last term in the decomposition above will not appear in the hydrodynamical equations while the gradient terms will give the dissipative terms. On times of order ε^{-2} the gradient modes have the effect of building up the dissipation in the system.

To get the hydrodynamical equations we start from the microscopic conservations laws

$$\frac{d}{dt}\varepsilon^d \sum_{x\in\Lambda} J(\varepsilon x) I_\beta(\eta_x) =$$

$$-\varepsilon^{d-2} \sum_{x\in\Lambda} \sum_{\alpha=1}^{d} J(\varepsilon x) \nabla^-_\alpha \left[g^\beta_{x,\alpha} + \sigma^\beta_{x,\alpha}\right] + M_t$$

where M_t is some martingale term and use the decomposition (5.16) to get

$$\frac{d}{dt}\varepsilon^d \sum_{x\in\Lambda} J(\varepsilon x) I_\beta(\eta_x) = \varepsilon^{d-2} \sum_{x\in\Lambda}\sum_{\alpha=1}^{d} J(\varepsilon x)\times$$

$$\nabla_\alpha^- \left[g_{x,\alpha}^\beta - \sum_{\gamma=0}^{d+1}\sum_{\vartheta=1}^{d} D_{\alpha,\vartheta}^{\beta,\gamma}\nabla_\vartheta I_\gamma + \mathcal{L}\tau_x F_\alpha^\beta \right] + M_t. \tag{5.17}$$

If we assume now that the non equilibrium state is close for ε small to the local equilibrium state by taking the expectation in (5.17) we would get the macroscopic equations for the hydrodynamical fields. The last term as explained before will vanish in the limit. The terms involving the gradients will give the dissipative terms (second order spatial derivatives). The terms coming from $g_{x,\alpha}^\beta$ are first order in the spatial derivatives so that they have a factor ε^{-1} in front. Again we face the problem of the control of waves propagating with large velocity. These waves can be controlled on this large time scale in the regime in which the fluid can be considered incompressible, as in the case of the Boltzmann equation.

We choose as initial state a local equilibrium with density, energy and velocity which are perturbations of order ε of constant profiles: the measure (5.4) with chemical potentials $n(x) = (n_0(x), \dots, n_{d+1}(x))$ slowly varying on microscopic scale and given by

$$n_\beta(x) = \lambda_\beta^{(0)} + \varepsilon\lambda_\beta^{(1)}(\varepsilon x) + \varepsilon^2\lambda_\beta^{(2)}(\varepsilon x), \tag{5.18}$$

for $\beta = 0, \dots, d+1$. Here $\lambda_\beta^{(k)}(z)$ are smooth functions on $\mathbb{T}_1^d$ for $k \geq 1$, $\lambda_\beta^{(0)} = 0$ for $\beta = 1, \dots, d$ and $\lambda_0^{(0)}$ and $\lambda_{d+1}^{(0)}$ constants. Correspondingly, as can be immediately checked, we have

$$\begin{aligned} m_0(\varepsilon x) &= \rho^{(0)} + \varepsilon\rho(\varepsilon x) + o(\varepsilon) \\ m_\beta(\varepsilon x) &= \varepsilon u_\beta(\varepsilon x) + o(\varepsilon), \quad \beta = 1, \dots, d \\ m_{d+1}(\varepsilon x) &= \mathcal{E}^{(0)} + \varepsilon\mathcal{E}(\varepsilon x) + o(\varepsilon). \end{aligned} \tag{5.19}$$

Assume that also at time t the local equilibrium, with chemical potentials of the form (5.18), with

$\lambda_\alpha^{(k)}$, $k \geq 1$, suitable functions of z and t, is a good approximation of the non equilibrium state. Then we get the following hydrodynamical equations:

$$
\begin{aligned}
&\mathrm{div} u = 0 \\
&\partial_t u_\beta + \partial_\beta p + K\, u \cdot \nabla u_\beta = \sum_{\alpha,\gamma=1}^{3} D_{\alpha,\alpha}^{\beta,\gamma} \partial_\alpha^2 u_\gamma \quad , \\
&\frac{\partial}{\partial t}\mathcal{E} + H u \cdot \nabla \mathcal{E} = \sum_{\alpha=1}^{3} \mathcal{K}_\alpha (\partial_\alpha^2 \mathcal{E})
\end{aligned}
\tag{5.20}
$$

where

$$
\mathcal{K}_\alpha = \chi + \Phi_1 \big(C \overline{D}_{\alpha,\alpha}^{0,0} - D_{\alpha,\alpha}^{4,0} \big) + \Phi_2 \big(\overline{D}_{\alpha,\alpha}^{4,4} - C D_{\alpha,\alpha}^{0,4} \big),
\tag{5.21}
$$

H, Φ_1, Φ_2, C are suitable constants depending on the details of the model and $\overline{D} = D - \chi I$. The density ρ is not independent of the energy: one can compute constants A, B such that

$$
A\rho + B\mathcal{E} = 0 \tag{5.22}
$$

This is the analog of the Boussinesq condition for this lattice model. First equation in (5.20) and (5.22) do not involve the time, so they have to be assumed at time $t = 0$ to avoid initial layers. Note the exact form of the second and third equation in (5.20) requires the choice of ω as a solution of an algebraic equation.

Remark. Above heuristic discussion does not follow what we did for Hamiltonian particle systems. The analog of that would be not to use the decomposition of the currents but rather write the conservation laws (5.17) in the form

$$
\begin{aligned}
&\frac{d}{dt}\varepsilon^d \sum_x J(\varepsilon x)[I_\beta(\eta_x) = \\
&- \varepsilon^{d-2} \sum_{x\in\Lambda} \sum_{\alpha=1}^{d} J(\varepsilon x) \nabla_\alpha^- w_{x,\alpha}^\beta + M_t.
\end{aligned}
$$

Instead of averaging with respect to the local equilibrium, we use a modified measure which takes into account also the correction to the local equilibrium. That amounts to assume that f_t is "close" to

$$\mu(\Phi) = Z_t^{-1} \exp\left\{ \sum_{x\in\Lambda} (n_\beta(\varepsilon x, t) I_\beta(\eta_x) + \varepsilon^2 \Phi(\eta) \right\}.$$

where n_β are given in (5.18) and

$$\Phi(\eta) = -\sum_{x\in\Lambda} \sum_{\alpha=1}^{d} \sum_{\beta=0}^{d+1} \partial_\alpha \lambda_\beta(\varepsilon x, t) \tau_x F_\alpha^\beta$$

for suitable functions F_α^β. It turns out that these functions have to be chosen to satisfy (5.16).

Using a method which is a combination of the two approaches, in [20] for the model $\mathcal{V}_2$ and in [2] for the thermal lattice gas, it is proven the following

Theorem 5.1

Assuming the finite volume ergodic property, the following is true:

(i) In dimension $d \geq 3$ the diffusion tensor does exist and is positive definite, bounded and $D \geq D_s$, $D \neq D_s$, where D_s is the diffusion tensor for the symmetric case. Moreover, D is explicitly given by the Green-Kubo formula

$$a \cdot (D\Theta) a = \chi a \cdot \Theta a + \int_0^\infty dt < a \cdot \sigma, P_t a \cdot \sigma >_0 \tag{5.23}$$

where P_t is the semigroup generated by $\mathcal{L}$ and Θ is the compressibility tensor

$$\Theta_{i,j}^{\alpha,\beta} = \delta_{i,j} \Big(E^\mu[I_\alpha I_\beta] - E^\mu[I_\alpha] E^\mu[I_\beta] \Big),$$
$$i, j = 1, \ldots d; \alpha, \beta = 0, \ldots, d+1$$

(ii) Let $d = 3$, and consider the thermal lattice gas. Let $u(z,t)$, $\mathcal{E}(z,t)$, $t \in [0, t_0]$, $t_0 > 0$ be any smooth classical solution of the equations (5.20), with D_α's and $\mathcal{K}_\alpha$'s given by (5.23) and (5.21). We start the process $\eta_t(x,v)$ with generator $\varepsilon^{-2}\mathcal{L}$ from the measure $\mu_{\Lambda,n}$ defined in (5.4), with chemical potentials $n_\alpha(x)$ of the form (5.18), satisfying

$$\mathrm{div}\lambda = 0, \quad A' \lambda_0^{(1)} + B' \lambda_4^{(1)} = 0$$

for some suitable constants A', B' depending on the model. We define the mass, momentum, energy and temperature empirical fields as

$$
\begin{aligned}
\nu_0^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x)\Big(I_0(\eta_x(t)) - \langle I_0\rangle\Big) \\
\nu_\beta^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x) I_\beta(\eta_x(t)),\ \ \beta = 1,2,3 \\
\nu_4^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x)\Big(I_4(\eta_x(t)) - \langle I_4\rangle\Big)
\end{aligned}
\tag{5.24}
$$

where $\langle\cdot\rangle$ stands for the average with respect to the reference measure μ^0. Then $(\nu_1(z,t),\ldots,\nu_3(z,t))$ and $(\nu)_4(z,t)$ converge, for $t \le t_0$, weakly in probability to $u(z,t)$ and $\mathcal{E}$. Moreover $\nu_0^\varepsilon(z,t)$ converge to $\rho(z,t)$.

We stress that the property $D \ge D_s$, $D \ne D_s$ can be interpreted as the fact that the anti-symmetric part of the dynamics, which is related to the velocity drift, is producing dissipation. That is what is expected to happens for Hamiltonian particle systems, where the Liouville operator is anti-symmetric and on times of order ε^{-2} it produces the dissipative terms in the Navier-Stokes equations. The contribution of the asymmetric motion to the heat conductivity $\mathcal{K}$ is given by a complicated combination of diffusion coefficients which we cannot check to be positive in general. This is due to the fact that the equation for the density in this model contains a diffusive part arising from the exclusion process and, as a consequence, the diffusion matrix has not the standard form. However, it has still some symmetry properties as a consequence of time reversal invariance that allows to prove the positivity at zero inverse temperature β. Moreover, we note that in the usual hydrodynamic equations $H = 1$. This is not the case in the present model.

We remark that the condition on the dimension is optimal in the sense that the transport coefficients are expected to be divergent in $d \le 2$ [22].

Also this aspect is shared by particle systems with the Boltzmann equation. Namely, the dimension of the velocity space has to be at least 3 in order that the transport coefficients make sense.

It should also be noted that for stochastic particle systems in $d \leq 2$ the typical size of the empirical fields would be the same or smaller than the fluctuations so that one cannot expect they to converge to deterministic variables. Anyway, it is worth to mention that above results cover also the 2-dimensional incompressible Navier-Stokes equation, because this equation does not describe the behavior of a really 2-dimensional fluid, but rather the one of a 3-dimensional fluid in correspondence of special initial data with planar symmetry. The Theorem 5.1 applies as well for such initial data and moreover, in this case, the result is global in time because the 2-dimensional incompressible Navier-Stokes equation has sooth solutions globally in time. The same remark of course applies to the derivation of the 2-dimensional incompressible Navier-Stokes equation from the Boltzmann equation.

A global result of derivation of the incompressible Navier-Stokes equation for general initial data, where smooth solutions are not known to exist uniquely, has been obtained for the model $\mathcal{V}_2$ in [37] where it is proved that the process converges by subsequences to a weak Leray-Hopf solutions of the incompressible Navier-Stokes equation. The lack of uniqueness of such solutions does not permit to identify the limit.

6. NON GRADIENT AND ENTROPY METHODS.

There are essentially two methods for proving the hydrodynamical limit for a lattice gas. We sketch in a really vague way the first one. It is based on some compactness property of the process. Given this property, one knows that the sequence of distributions of the empirical measure P^ε is convergent, up to subsequences, to some P^0. Then, one shows that P^0 is supported on the weak solutions of the limiting equation. Finally, if for that equation it is possible to prove a uniqueness theorem then the empirical measure converges weakly to that solution. In this one can construct a solution to the hydrodynamical equation as limit of a sequence of empirical measures which satisfy the corresponding local conservations laws studied as martingale problems. This approach has been proposed in [GPV] and widely used in many reversible and non reversible models. Stronger statements in this diretions, as large deviations estimates have been obtained, for example in the already cited paper [37] where it is proved the convergence of the model $\mathcal{V}_2$ to the set of the weak solutions to the incompressible Navier-Stokes equation.

Theorem 6.1 *Let P^ε be the distributions of the empirical momentum densities (5.24). Then P^ε are tight as a set of probability measures on the space $L^2(0,T;H_{-2})$ with the weak topology and any weak limit P is supported entirely on weak solutions of the incompressible Navier-Stokes equations for this model. Moreover, these weak solutions satisfy an energy inequality.*

The second method is based on the study of the relative entropy between the non equilibrium measure and the local equilibrium measure. The method requires smoothness of the solutions of the limiting problem. Given a classical solution of the limiting equation then one proves that the relative entropy of the nonequilibrium density with respect the local equilibrium vanishes in the limit, provided that the chemical potentials are suitably chosen in terms of this solution. That is enough to obtain the convergence of the empirical measures to the corresponding limits. The theorem stated in Section 5 is of this form. Results of this form are similar to those mentioned in Section 2 for the hydrodynamical limit of the Boltzmann equation while the results in [35], [42], [1] are more in the spirit of Theorem 6.1.

By the entropy method it is not possible to get the limit for times such that the limiting solution develops singularity. Anyway, the only limitations on the time of validity of the result depend on the possible growth of singularities of the limiting equations, not on the properties of the microscopic system. On the other hand, one gets not only the convergence but also the control of the relative entropy, which ensures that the non equilibrium measure is accurately approximated, in an L_1-norm, up to the lowest order in ε, by the local equilibrium.

A crucial step in order to apply both methods is the slow-fast modes decomposition of the currents.

In this section we explain first the main points of the strategy for proving such a decomposition and then we discuss the entropy method for the lattice gas model with collisions.

Slow-fast modes decomposition.

We explain the strategy in the case of the lattice gas model described in the previous section: particles with velocity $v \in \mathcal{V}$ jump on the lattice in such a way that in any site there are not two particles with the same velocity and collide by means of "elastic collisions". To show the slow-fast modes decomposition we use a kind of geometric argument, namely we construct a Hilbert space of local functions and show that, roughly speaking, it decomposes in the direct sum of a subspace spanned by the gradients ∇I_α of the conserved quantities and a subspace generated by functions of the form $\mathcal{L}g$.

Let μ^n be the Gibbs measure with chemical potentials $n = \{n_\alpha, \alpha = 0, \ldots, d+1\}$. The marginal in Ω_Λ is

$$\mu_\Lambda^n = Z_{n,\Lambda}^{-1} \prod_{x\in\Lambda} \exp\Big\{\sum_{\beta=0}^{d+1} n_\beta I_\beta(\eta_x)\Big\}$$

Let us recall the definition of the space $\mathcal{G}$ given in the previous section. Fix a vector $\bar{n} \in \mathbf{R}^{d+2}$ of chemical potentials and define $\mathcal{G}$ be the space of local functions of η satisfying

$$\mathbb{E}^{\mu^{\overline{n}}}[g] = 0, \qquad \sum_{x\in\mathbb{Z}^d} \mathbb{E}^{\mu^{\overline{n}}}[g; I_\alpha(\eta_x)] = 0, \qquad \alpha = 0, \ldots, d+1. \tag{6.1}$$

Here $\mathbb{E}^\mu[g;f] = \mathbb{E}^\mu\Big[(g - \mathbb{E}^\mu[g])(f - \mathbb{E}^\mu[f])\Big]$. Let $\mu_m = \mu^{n(m)}$ be the product measure such that $\mathbb{E}^{\mu_m}[I_\alpha] = m_\alpha$ for $\alpha = 0, \ldots, d+1$ and $\hat{g}(m) = \mathbb{E}^{\mu_m}[g]$. Then the second condition in (6.1) is equivalent to the second condition (5.15) stated in Section 5:

$$\frac{\partial \hat{g}(m)}{\partial m_\alpha}\Big|_{m=\overline{m}} = 0, \quad \alpha = 0, \ldots, d+1. \tag{6.2}$$

where $\overline{m}_\alpha = \mathbb{E}^{\mu^{\overline{n}}}[I_\alpha]$, $\alpha = 0, \ldots, d+1$, are the values corresponding to the equilibrium measure $\mu^{\overline{n}}$. From now on we drop the overline, because n will be fixed.

To understand the meaning of this definition, let us consider the space of the local function on Ω, i.e.

the functions depending only on the configuration in a finite number of sites of Ω and introduce the scalar product of two local functions f, g, as

$$< f, g >_0 = \sum_{x \in \mathbb{Z}^d} \mathbb{E}^{\mu^n}[\tau_x f; g].$$

Recall also that the null space of the generator $\mathcal{L}$, by the finite volume ergodic property, is spanned by the conserved quantities I_α, $\alpha = 0, \ldots, d+1$.

We define the *projector on the null space of* $\mathcal{L}$, as

$$\mathcal{P}g = \sum_{\alpha=0}^{d+1} < g, \hat{I}_\alpha > \hat{I}_\alpha \tag{6.3}$$

where $\hat{I}_\beta$ are combinations of I_a that forms an orthonormal set with respect $< \cdot, \cdot >_0$. Define the $(d+2) \times (d+2)$ compressibility matrix $\bar{\theta}$ as

$$\bar{\theta}_{\alpha,\beta} = < I_\alpha, I_\beta >_0 .$$

Then $\hat{I} = \bar{\theta}^{-1/2}(I - \mathbb{E}^{\mu^n}[I])$.

The space $\mathcal{G}$ defined before is the space of the local functions having no component on the null space of $\mathcal{L}$. Hence a local function $g \in \mathcal{G}$ is such that $\mathcal{P}g = 0$.

We observe that any local function of the form $\mathcal{L}f$ belongs to this space because $E^{\mu^n}[I_\alpha; \mathcal{L}f] = 0$. Moreover, because of the sum over the spatial translations, also functions of the form ∇I_α belong to this space. These functions are indeed special because their norm $< \nabla I_\alpha, \nabla I_\alpha >_0$ is zero.

To construct the Hilbert space where the equation (5.16) will be solved, we need to define a new scalar product. It is convenient to choose it so that the functions ∇I_α and the functions of the form $\mathcal{L}f$ are orthogonal. Since for any local function g we have $< \nabla I_\alpha, g >_0 = 0$ we also have, formally, $< \nabla I_\alpha, \mathcal{L}^{-1}\mathcal{L}g >_0 = 0$. We cannot give sense to $\mathcal{L}^{-1}$, but we can instead construct a scalar product using the inverse of the symmetric generator $\mathcal{L}_s^{-1}$. To this end we start constructing it in finite volumes and then extend the construction to the infinite volume. Let B be a finite subset of $\mathbb{Z}^d$ and $|B|$ its cardinality. The empirical averages of the conserved quantities in the block B, $\overline{I}_B$ are defined as

$$\overline{I}_{\beta,B}(\eta) = \frac{1}{|B|} \sum_{x \in B} I_\beta(\eta_x), \quad \beta = 0, \ldots, d+1 \tag{6.4}$$

For any finite subset B of $\mathbb{Z}^d$ and for any vector $m \in \mathbf{R}^{d+2}$, we define the *canonical* Gibbs measure $\mu_{B,m}$, as the uniform probability distribution on the set $\Omega_{B,m}$ defined as

$$\Omega_{B,m} = \{\eta \in \Omega_B \mid \overline{I}_B(\eta) = m\},$$

i.e. the set of configurations on the block B such that the invariants have the fixed values $m|B|$.

Fixed B, for any local function G, set

$$\tilde{G} = G - \mathbb{E}^{\mu_{B,\overline{I}_B}}[G] \tag{6.5}$$

We call $\mathcal{L}_{s,B}$ the symmetric part of the generator $\mathcal{L}$ restricted to the block B, meaning that only jumps over bounds inside B and collisions in points of B are allowed. By the finite volume ergodic property we know that the dynamics with generator $\mathcal{L}_{s,B}$ in the finite volume B is a transitive Markov process, because it is ergodic on the set of configurations η such that $\overline{I}_{\beta,B}$ $(\beta = 0, \ldots, d+1)$ are fixed. Hence, $\mathcal{L}_{s,B}^{-1}\tilde{G}$ is well defined by the definition (6.5) of $\tilde{G}$.

Given any local function G on Ω_B, we define the finite volume "variance" $V_B(G,m)$ as

$$V_B(G,m) = \frac{1}{|B|}\langle\Big[\sum_{|x|\in B}(\tau_x\tilde{G}\Big](-\mathcal{L}_{s,B})^{-1}\Big[\sum_{|x|\in B}(\tau_x\tilde{G}\Big]\rangle_{\mu_{B,m}} \tag{6.6}$$

and the "variance" $V(G,m)$ of G as

$$V(G,m) = \limsup_{B\to\infty} V_B(G,m) \tag{6.7}$$

Define the semi-norm on $\mathcal{G}$

$$\|\cdot\|_{-1}^2 = V(\cdot)$$

where $V(\cdot)$ is the infinite volume variance defined by (6.7), with the m dependence suppressed. Let

$$\mathcal{G}^0 = \{\sum_{\alpha=0}^{d+1}\sum_{j=1}^{d} a_{\alpha,j}\nabla_j I_\alpha(\eta_0),\ a_{\alpha,j} \in \mathbb{R}\}$$

be the space of the gradients of the conserved quantities (remember that η_0 is the configuration in $x = 0$). Here, for any function g,

$$\nabla_j g(\eta) := g(\tau_{e_j}\eta) - g(\eta).$$

We introduce the following equivalence relation: two elements of $\mathcal{G}$ are equivalent if they differ by an element of $\mathcal{G}^0$. The quotient of $\mathcal{G}$ with respect to this equivalence relation will be denoted by $\mathcal{G}\backslash\mathcal{G}^0$.

In [2] it is stated (the proof is essentially already in [20], where the approach of [32] is used) the following theorem

Theorem 6.2 *For all $g \in \mathcal{G}$, one has $\| g \|_{-1} < \infty$. Furthermore, let $\overline{\mathcal{G}}$ denote the closure of $\mathcal{G}$ under the semi-norm $\| \cdot \|_{-1}$ and define*

$$\ll g, h \gg = \frac{1}{4}\big[\, \| g + h \|_{-1} + \| g - h \|_{-1} \,\big]. \qquad (6.8)$$

Then $\ll \cdot\,, \cdot \gg$ is an inner product and $\overline{\mathcal{G}}$ equipped with this inner product is a Hilbert space. Moreover $\overline{\mathcal{G}}$ can be decomposed as

$$\overline{\mathcal{G}} = \overline{\mathcal{L}_s[\mathcal{G}\backslash\mathcal{G}^0] \oplus \mathcal{G}^0} := \mathcal{H}. \qquad (6.9)$$

Hence, up to some limiting procedure, we have the decomposition of $\mathcal{G}$ in a “gradient” and a *fast decaying* part $\mathcal{L}_s f$. A byproduct of Theorem 6.2 is that we can give rigorous sense to the formal expression $\langle f, \mathcal{L}_s^{-1} h\rangle_0$. In fact, the inverse of $\mathcal{L}_s$ is well defined only on a space with fixed number of particles, momentum and energy (canonical ensemble) and one is forced to work at finite volume and take the limit afterwards. In Theorem 6.2 instead we have an infinite volume formulation, which makes possible the geometric argument below.

The currents w_β^α, are the main object we want to decompose in terms of gradients and fast modes. They are not in $\mathcal{G}$. We introduce the modified currents $\sigma_\alpha^\beta \in \mathcal{G}$ as

$$\sigma_\beta^\alpha = w_\beta^\alpha - \mathcal{P} w_\beta^\alpha, \quad \sigma_\beta^{*\alpha} = w_\beta^{*\alpha} - \mathcal{P} w_\beta^{*\alpha} \qquad (6.10)$$

where $\mathcal{P}$ is the projector (6.3).

We state now some properties of the scalar product $\ll \cdot, \cdot \gg$ that we will use below. For any $g, h \in \mathcal{G}$:

$$\ll \mathcal{L}_s g, h \gg = -\big\langle h, g \big\rangle_0, \qquad (6.11)$$

$$\mathcal{G}^0 \perp \mathcal{L}_s\mathcal{G}, \tag{6.12}$$

$$\begin{aligned} -\left\langle g, w_j^{*\ \alpha}\right\rangle_0 &= -\left\langle g, \sigma_j^{*\ \alpha}\right\rangle_0, \\ -\left\langle g, w_j^{\alpha}\right\rangle_0 &= -\left\langle g, \sigma_j^{\alpha}\right\rangle_0 \end{aligned} \tag{6.13}$$

$$\ll \sigma_j^{\beta}, \nabla_i I_\alpha(\eta) \gg = \delta_{i,j}\delta_{\alpha,\beta}\gamma V(\nabla_i I_\alpha), \tag{6.14}$$

for $\alpha, \beta = 1, \ldots, d$ and $i, j = 1, \ldots, d$.

$$\ll \sigma_i^0, \nabla_j I_0(\eta) \gg = \delta_{i,j}\gamma V(\nabla_i I_0), \quad i, j = 1, \cdots, d.$$

Theorem 6.2 allows to decompose the currents σ, up to some approximation procedure, in a term ϕ in $\mathcal{G}^0$ which is a linear combination of gradients of I_α and a term π of the form $\mathcal{L}_s g$ for a function g in $\mathcal{L}_s[\mathcal{G}\backslash\mathcal{G}^0]$. Moreover, ϕ is orthogonal to π in the sense that $\ll \phi, \pi \gg = 0$.

We note that we have put in italic the words " fast decaying" after the statement of Theorem 6.2 because the really fast modes are those of the form $\mathcal{L}f$. $\mathcal{L}_s f$ would be fast decaying modes if the dynamics where generated by $\mathcal{L}_s$, rather than $\mathcal{L}$. Theorem 6.2 is an intermediate step toword the proof of the decomposition we are looking for. From Theorem 6.2 we get that there exists a unique tensor B such that

$$\sigma_i^\alpha + \sum_{\beta=0}^{d+1}\sum_{j=1}^{d} B_{i,j}^{\alpha,\beta}\nabla_j I_\beta \in \overline{\mathcal{L}_s\mathcal{G}}$$

Therefore, the components of σ in $\mathcal{G}^0$ are determined by finding a suitable tensor B such that

$$\ll \sigma_i^\alpha + \sum_{\beta=0}^{d+1}\sum_{j=1}^{d} B_{i,j}^{\alpha,\beta}\nabla_j I_\beta, \nabla_k I_\vartheta \gg = 0$$

for any $\alpha, \vartheta = 0, \cdots d+1$, $i, k = 1, \ldots, d$. Hence

$$B_{i,j}^{\alpha,\beta} = \sum_{\theta=0}^{d+1}\sum_{k=1}^{d} \ll \nabla_j I_\beta, \nabla_k I_\vartheta \gg^{-1} \ll \sigma_i^\alpha, \nabla_k I_\vartheta \gg$$

Note that by (6.14), $\ll \sigma_i^\alpha, \nabla_j I_\beta \gg = \gamma\delta_{i,j}\bar{\theta}_{\alpha,\beta}$ and by (6.13), $\ll \sigma_i^\alpha, \nabla_j I_\beta \gg = \gamma \ll \nabla_i I_\alpha, \nabla_j I_\beta \gg$. Therefore, $B_{i,j}^{\alpha,\beta} = \gamma\delta_{\alpha\beta}\delta_{i,j}$, so that the gradient part in this decomposition is exactly the symmetric current. The same argument, for the non symmetric case, will provide the diffusion coefficients.

The main step in proving Theorem 6.2 is finding a function g such that

$$||\sigma^{\perp} - \mathcal{L}_s g||_{-1} = 0 \tag{6.15}$$

were $\sigma^{\perp}$ is the component of σ orthogonal to the space of gradients $\mathcal{G}^0$. The function g in (6.15) is determined by solving the resolvent equation for $\mathcal{L}_s$. This has been done in [20], using techniques introduced in [32] and [41].

To get the right decomposition in the non reversible case it is important to solve the resolvent equation for the full generator $\mathcal{L}$. This result requires to know some properties of the associated Green function. The difficulty is that the asymmetric generator is not a small perturbation of the symmetric one and one needs to use more refined arguments to estimate the Green function $(\lambda - \mathcal{L})^{-1}$.

In next Theorem we recall the main estimates on the solution of the resolvent equation

Theorem 6.3 . *Let H_k for $k = -1, 0, 1$ the completion of the spaces*

$$\mathcal{H}_k = \{f \in \mathcal{G} : ||f||_k^2 :=< f, (-\mathcal{L}_s)^k f >_0 < \infty\}.$$

with respect to the corresponding norms (for $k = -1$ this has to be understood in terms of the variance (6.7)).

The solution u_λ of the resolvent equation

$$\lambda u_\lambda - \mathcal{L} u_\lambda = h$$

for $d \geq 3$ satisfies uniformly in λ the estimates

$$||u_\lambda||_1 < \infty, \quad \lambda ||u_\lambda||_{-1} < \infty$$

This Theorem has been proven in proven in [20] for the model $\mathcal{V}_2$ but the proof is easily extended to the model introduced in the previous sections, provided the finite volume ergodic property is satisfied. In fact the proof depends only on the properties of the symmetric simple exclusion (see also [32]) and on the following properties of the collision operator: the collisions are local, binary and conserve only the quantities I_α's.

We summarize some of the properties of the solution of the resolvent equation, that we will need below, in next Theorem.

Theorem 6.4 . *Let $h \in \mathcal{G}$. Then*

(i) *For any $\varepsilon > 0$ there exists a local function $u_\varepsilon \in \mathcal{G}$ such that*
$$||\mathcal{L}u_\varepsilon - h||_{-1} \le \varepsilon$$

(ii) *u_ε converges strongly in $\mathcal{H}_1$ to some $g \in \mathcal{H}_1$*

(iii) *u_λ converges strongly in H_1 to the same $g \in \mathcal{H}_1$*

(iv) $\lim_{\lambda\to 0} ||\lambda u_\lambda||_0 = 0$

Below, for the function g whose existence is guaranteed by above theorem we will use the notation

$$g = \mathcal{L}^{-1}h.$$

Proof. The proof of these statements can be found in [20], [41], [32]. We report the main steps for sake of completeness.

By Theorem 6.3 u_λ and λu_λ converge by subsequences weakly to some $g \in \mathcal{H}_1$ and to 0 in $\mathcal{H}_{-1}$ respectively. Then, by the resolvent equation, $-\mathcal{L}u_\lambda$ converges weakly to h in $\mathcal{H}_{-1}$. This implies that there are convex combinations u_ε of u_λ's such that u_ε and $-\mathcal{L}u_\varepsilon$ converge strongly to g and h in $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ respectively and that proves (i) and (ii). Also, $< u_\varepsilon, (-\mathcal{L})u_\varepsilon >_0$ converges to $< g, h >_0$, which implies $< g, h >_0 = ||g||_1^2$. By using the resolvent equation and the fact that g is the weak limit of u_λ we get

$$||g||_1^2 \le \lim_{\lambda\to 0}[\lambda||u_\lambda||_0^2 + ||u_\lambda||_1^2] \le < g, h >_0$$

This proves (iii) and (iv).

Finally, the following Theorem on the structure of the space $\mathcal{G}$ has been proven in [20], [2]

Theorem 6.5 .

(i) $\overline{\mathcal{L}\mathcal{G} + \mathcal{G}_0} = \overline{\mathcal{G}} = \overline{\mathcal{L}^*\mathcal{G} + \mathcal{G}_0}$

(ii) Let
$$\mathcal{G}_w = \Big\{\sum_{\alpha=0}^{d+1}\sum_{j=1}^{d} b_j^\alpha \sigma_j^\alpha,\ b_j^\alpha \in \mathbf{R}\Big\}.$$

Then
$$\begin{aligned} &\overline{\mathcal{G}_w + \mathcal{L}_s\mathcal{G}} = \overline{\mathcal{G}} = \overline{\mathcal{G}_w + \mathcal{L}\mathcal{G}}\ ; \\ &\overline{\mathcal{G}_{w^*} + \mathcal{L}^*\mathcal{G}} = \overline{\mathcal{G}}. \end{aligned} \tag{6.16}$$

The decomposition of the space $\mathcal{G}$ in Theorem 6.5 is not in the form of direct sum of orthogonal subspaces so that we cannot use the procedure we used before to find the tensor B (that is also the procedure suitable for a reversible non-gradient dynamics). In order to reduce to that, we introduce a map T on $\mathcal{G}$ which transforms $\overline{\mathcal{G}_0}$ in a space orthogonal to $\overline{\mathcal{L}^*\mathcal{G}}$.

Theorem 6.6 .

(i) Let $u_j^\beta = \gamma \nabla_j I_\beta$. Put $a \cdot d = \sum_{j\geq 1, \beta \geq 0} a_j^\beta d_j^\beta$. Define T to be the linear transformation from $\mathcal{G}$ to $\mathcal{G}$ that, on the functions of the form $b \cdot \sigma + \mathcal{L} g$ acts as

$$T(b \cdot \sigma + \mathcal{L} g) = b \cdot u + \mathcal{L}_s g. \tag{6.17}$$

Then T has norm bounded above by 1, hence can be extended by continuity to $\overline{\mathcal{G}}$. Moreover

$$(a) \quad T\nabla_j I_\alpha(\eta) \perp \mathcal{L}^*\mathcal{G} \quad \text{for } j \geq 1, \alpha \geq 0$$

$$(b) \quad \ll T\nabla_i I_\alpha(\eta)\, ,\, \sigma_j^{*\ \beta} \gg = \Theta_{i,j}^{\alpha,\beta}$$

(ii) Let the diffusion tensor D be defined as the unique tensor such that

$$\sigma_i^\alpha - \sum_{\beta=0}^{d+1} \sum_{j=1}^{d} D_{i,j}^{\alpha,\beta} \nabla_j I_\beta \in \overline{\mathcal{L}\mathcal{G}}$$

Then the elements $D_{i,j}^{\alpha,\beta}$ are given by

$$D_{i,j}^{\alpha,\beta} = (\Theta M^{-1})_{i,j}^{\alpha,\beta} \tag{6.18}$$

where M is the tensor

$$M_{i,j}^{\alpha,\beta} = \ll \nabla_i I_\alpha(\eta), T\nabla_j I_\beta(\eta) \gg .$$

As a quadratic form, $D \geq D_s$ and $D \neq D_s$, where $(D_s) = \gamma \Theta \delta_{i,j}$. Furthermore, assuming the dimension $d = 3$, and the thermal lattice gas, there exist constants $D_k, k = 1, \cdots, 6$ such that

$$D_{j,k}^{\alpha,\beta} = \delta_{j,k}\delta_{\alpha,\beta}[D_1 + D_2\delta_{\alpha,j}] + D_3\delta_{\alpha,j}\delta_{\beta,k} + D_6\delta_{j,\beta}\delta_{k,\alpha}. \tag{6.19}$$

for $\alpha, \beta = 1, 2, 3,$

$$D^{0,0}_{i,j} = \delta_{i,j} D_0, \quad D^{4,4}_{i,j} = \delta_{i,j} D_4$$

$$D^{\alpha,\beta}_{i,j} = 0 \text{ if } \begin{cases} \alpha = 0, 4 \text{ and } \beta = 1, 2, 3 \\ \text{or} \\ \alpha = 1, 2, 3 \text{ and } \beta = 0, 4. \end{cases} \tag{6.20}$$

and

$$(\Theta^{-1} D)^{0,4}_{i,j} = (\Theta^{-1} D)^{4,0}_{i,j} = \delta_{i,j} D_5 \tag{6.21}$$

Eq. (6.19) and (6.20) are consequence of the invariance properties under spatial rotations and reflection of the dynamics and hence of the matrix M, as shown in [20]. Eq. (6.21) instead is consequence of the time-reversal symmetry.

Variational formulas for the diffusion coefficients D and its inverse D^{-1} can be obtained. They are very useful in proving for example positivity and boundedness properties. We refer to [20] and [31], where similar formulas are proven and discussed for the $\mathcal{V}_2$ simple exclusion respectively. They are summarized in the following

Theorem 6.7 : *The following variational representation for the diffusion matrix D holds: For any* $a = \{a^\alpha_j \in I\!R, \alpha = 0, \ldots, d+1, j = 1, \ldots, d\}$

$$a \cdot (D\Theta) a = \inf_{h \in \mathcal{G}} \ll \sigma \cdot a + \mathcal{L}h, \sigma \cdot a + \mathcal{L}h \gg, \tag{6.22}$$

$$a \cdot D\Theta^{-1} a = \inf_{h \in \mathcal{G}} \ll \nabla I \cdot a + \mathcal{L}h, \nabla I \cdot a + \mathcal{L}h \gg . \tag{6.23}$$

Now we sketch the proof of the Green-Kubo formulas for the transport coefficients. We start with a Lemma providing an expression of D in terms of the strong limit in $\mathcal{H}_1$ of the sequence u_ε introduced in Theorem 6.4. The proof is based on the formal argument given in [19] (at the end of Section 5) and on Theorem 6.4.

Lemma 6.8 . *The diffusion matrix* $\bar{D} = D - \gamma\, I\!I$ *satisfies*

$$a \cdot (\bar{D}\Theta) a = \lim_{\varepsilon \to 0} ||a \cdot u_\varepsilon||^2_1 \tag{6.24}$$

Proof.

The currents σ^α_β and $\nabla_\alpha I^\beta$ belong to the space $\overline{\mathcal{G}}$. The diffusion coefficient is found as the matrix such that

$$\sigma + D\nabla I \in \overline{\mathcal{L}\mathcal{G}}$$

Hence there exists $g \in \overline{\mathcal{L}\mathcal{G}}$ such that

$$||\sigma + D\nabla I - g||_{-1} = 0$$

The function g may be non-local but can be approximated by local functions $u_\varepsilon \in \mathcal{L}\mathcal{G}$ such that, by the Theorem 6.5,

$$\sigma + D\nabla I - \mathcal{L}u_\varepsilon = h_\varepsilon, \quad \lim_{\varepsilon \to 0} ||h_\varepsilon||_{-1} = 0. \tag{6.25}$$

D is given by the expression (6.18)

$$a \cdot (D^{-1}\Theta)a = \ll a \cdot \nabla I, Ta \cdot \nabla I \gg .$$

By (6.25) we have $\nabla I = D^{-1}[\sigma - \mathcal{L}u_\varepsilon - h_\varepsilon]$. By the definition of the map T we have that $T(\sigma - \mathcal{L}u_\varepsilon) = \gamma\nabla I - \mathcal{L}_s u_\varepsilon$. Hence

$$\begin{aligned} a \cdot (D^{-1}\Theta)a = \ll a \cdot \ & D^{-1}[\sigma - \mathcal{L}u_\varepsilon - h_\varepsilon], \\ & a \cdot D^{-1}[\gamma\nabla I - \mathcal{L}_s u_\varepsilon - Th_\varepsilon] \gg \end{aligned}$$

which implies, using the symmetry of the matrices $D^{-1}\Theta$ and Θ,

$$a \cdot (D\Theta)a = \gamma \ll a \cdot (\sigma - \mathcal{L}u_\varepsilon), a \cdot (\chi\nabla I + \mathcal{L}_s u_\varepsilon) \gg + R_\varepsilon$$

with R_ε vanishing when ε goes to zero.

It is easy to see by using the properties of the scalar product (6.11)–(6.14) that

$$\begin{aligned} \ll a \cdot (\sigma - \mathcal{L}u_\varepsilon), a \cdot (\gamma\nabla I - \mathcal{L}_s u_\varepsilon) \gg = \\ \gamma a \cdot \Theta a + \ll a \cdot \mathcal{L}u_\varepsilon, a \cdot \mathcal{L}_s u_\varepsilon \gg . \end{aligned}$$

Hence

$$a \cdot (D\Theta)a = \gamma a \cdot \Theta a + \ll a \cdot \mathcal{L}u_\varepsilon, a \cdot \mathcal{L}_s u_\varepsilon \gg + R_\varepsilon$$

By the property (6.11) of the scalar product we have

$$
\begin{aligned}
\ll a \cdot \mathcal{L} u_\varepsilon, a \cdot \mathcal{L}_s u_\varepsilon \gg &= - < a \cdot \mathcal{L} u_\varepsilon, a \cdot u_\varepsilon >_0 \\
&= < a \cdot u_\varepsilon, -\mathcal{L}_s a \cdot u_\varepsilon >_0 \\
&= ||a \cdot u_\varepsilon||_1^2 .
\end{aligned}
$$

Since $R_\varepsilon \to 0$ we have proved the Lemma.

Theorem 6.9 *The following Green-Kubo representation for the transport matrix D holds:*

$$
a \cdot D\Theta a = \gamma a \cdot \Theta a + < \sigma \cdot a, (-\mathcal{L})^{-1} \sigma \cdot a >_0 \quad (6.26)
$$

Proof. By iii) of Theorem 6.9 we have that $\lim_{\lambda \to 0} ||u_\lambda||_1 = \lim_{\varepsilon \to 0} ||u_\varepsilon||_1$. On the other hand, u_λ is the solution of the resolvent equation $\lambda u_\lambda - \mathcal{L} u_\lambda = \sigma$ so that

$$
\begin{aligned}
\lambda ||a \cdot u_\lambda||_0^2 - ||a \cdot u_\lambda||_1^2 &= < a \cdot \sigma, a \cdot u_\lambda >_0 \\
&= < a \cdot \sigma, (\lambda - \mathcal{L})^{-1} a \cdot \sigma >_0 .
\end{aligned}
$$

By iii) and iv) of Theorem 6.9 $\lambda ||u_\lambda||_0 \to 0$ and $||u_\lambda||_1 \to ||g||_1$ for some g in H_1. Hence

$$
\lim_{\lambda \to 0} ||a \cdot u_\lambda||_1^2 = < a \cdot \sigma, (-\mathcal{L})^{-1} a \cdot \sigma >_0 = ||a \cdot g||_1^2 < \infty
$$

Finally by ii) of Theorem 6.9 we have that

$$
\begin{aligned}
a \cdot (\bar{D}\Theta) a &= \lim_{\varepsilon \to 0} ||a \cdot u_\varepsilon||_1^2 = \lim_{\lambda \to 0} ||a \cdot u_\lambda||_1^2 \\
&= < a \cdot \sigma, (-\mathcal{L})^{-1}, a \cdot \sigma >_0 .
\end{aligned}
$$

For any fixed $\lambda > 0$ we can write

$$
< f, (\lambda - \mathcal{L})^{-1} f >_0 = \int_0^\infty dt < f, P_t(\lambda) f >_0
$$

where $P_t(\lambda)$ is the semigroup generated by $\mathcal{L} - \lambda$. By using the theorem of dominated convergence we get in the limit $\lambda \to 0$ that

$$
< f, (-\mathcal{L})^{-1} f >_0 = \int_0^\infty dt < f, P_t f >_0
$$

In conclusion we get the Green-Kubo formula for $\bar{D}$

$$a \cdot (\bar{D}\Theta)a = \int_0^\infty dt < a \cdot \sigma, P_t a \cdot \sigma >_0 . \qquad (6.27)$$

The diffusion coefficient has been defined in different ways depending on the point of view adopted. Spohn [39] proposed a definition based on the long time behavior of the structure function. Adapting its definition to our model we introduce the coefficient $D^{(1)}$ as

$$(D^{(1)}\Theta)_{i,j}^{\alpha,\beta} = \lim_{t\to\infty} \frac{1}{t}\Big\{ \sum_x x_i x_j E\Big[I_\alpha(x,t); I_\beta(0,0)\Big] - \sum_{\gamma,\delta=0}^{4} (v_i^{\alpha,\gamma} t)\bar{\theta}^{\gamma\delta}(v_j^{\delta,\beta} t)\Big\}$$

where v_i is the tensor $v_i^{\alpha,\beta}$ defined as

$$(\bar{\theta} v_i)^{\alpha,\beta} = \sum_x x_i E\Big[I_\alpha(x,t); I_\beta(0,0)\Big]$$

and $\bar{\theta}$ is the compressibility matrix. This is the diffusion coefficient that should appear in the equation for the equilibrium fluctuations of density, momentum and energy under the assumption that they are Gaussian.

On the other hand the linear response theory gives the transport coefficients expressed in terms of the Green-Kubo formulas. Call $D^{(2)}$ the transport matrix as given by the Green-Kubo formula

$$(D^{(2)}\Theta)_{i,j}^{\alpha,\beta} = \gamma\Theta_{ij}^{\alpha,\beta} + \int_0^\infty dt < \sigma_i^\alpha, P_t a \cdot \sigma_j^\beta >_0$$

Finally, there is the transport matrix D that appears in the Navier-Stokes equations given by (6.18). One expects that all these definitions be related. In [31] it is discussed this problem for the asymmetric simple exclusion process and it is shown that $(D)^{(1)}$ can be written also as

$$(D)_{i,j}^{(1)} = \delta_{i,j} + \lim_{t\to\infty} \frac{1}{2t\ \Theta} \int_0^t ds \int_0^t ds' \Big[< \sigma_i^\alpha, P_{s'}\sigma_j^\beta >_0 + < \sigma_j^\alpha, P_{s'}\sigma_i^\beta >_0 \Big]$$

with the compressibility tensor Θ a scalar in that case, which is the natural form for the diffusion coefficient in the equation for the equilibrium fluctuation of the density field. For our model one can prove, following the argument in [31] and [39], that

$$(D^{(1)}\Theta)_{i,j}^{\alpha,\beta} = \gamma\Theta_{i,j}^{\alpha,\beta} + \lim_{t\to\infty}\frac{1}{t}\int_0^t ds\int_0^s ds' < \sigma_i^\alpha, P_t\sigma_j^\beta >_0$$

Note that in our case $D\Theta$ is symmetric separately in the indices α, β and i, j. Since we have shown above that the time integral in (6.26) is finite, we can conclude that $D^{(1)} = D^{(2)}$. Moreover, we have already shown that $D = D^{(2)}$, so that we can conclude the equivalence of all definitions.

Entropy method

The entropy method was proposed in [44] for proving the hydrodynamical limit for a Ginsburg-Landau model. The general idea is to compare the non-equilibrium measure with the local equilibrium modeled according to the behavior of the system on the hydrodynamic space-time scales with parameters slowly varying in space and suitably chosen in such a way to reproduce the correct hydrodynamical fields in the limit. The tool we use to compare two probability measures μ and ν is the relative entropy of μ with respect to ν defined as $+\infty$ if μ is not absolutely continuous with respect to ν and

$$S(\mu|\nu) = \int d\nu f \log f$$

where $f = d\mu/d\nu$ is the Radon-Nykodim derivative. Alternatively, $S(\mu|\nu)$ can be defined as the smallest constant K such that for any bounded function h

$$\int h d\mu - \log\int e^h d\nu \le K$$

This second definition, equivalent to the first, implies the entropy inequality for any positive γ:

$$\int h d\mu \le \frac{1}{\gamma}S(\mu|\nu) + \frac{1}{\gamma}\log\int d\nu e^{\gamma h}$$

This inequality implies that for any measurable set A

$$P_\mu(A) \leq \frac{\log 2 + S(\mu|\nu)}{\log\left(1 + P_\nu(A)^{-1}\right)}$$

Therefore, if $P_\nu(A)$ is small and the relative entropy is bounded also $P_\mu(A)$ is small. We remark that the relative entropy is zero if and only if $\mu = \nu$ and $\|f - 1\|^2_{L_1(d\nu)}$ is a lower bound for the relative entropy. Hence the relative entropy gives an estimate of how close are two measures.

We will explain how this method works for the model introduced in Section 5. Assume that at time zero the density f_0 of the system in Λ is the Gibbs measure (5.4):

$$\mu^n_\Lambda(\eta) = Z^{-1}_{n,\Lambda} \prod_{x\in\Lambda} \exp\Big\{\sum_{\beta=0}^{d+1} n_\beta I_\beta(\eta_x)\Big\} \qquad (6.28)$$

with n given by (5.18). We use the reference measure μ^0 given by (5.5) and denote by f_t also the non equilibrium density with respect to μ^0. The specific entropy (namely the entropy per site) of the initial measure with respect to μ^0 is of order ε^2. Since it can only decrease in time, it is bounded by ε^2 at later times. Therefore, if we replace μ^0 with a suitable local equilibrium, we get a better approximation to the non equilibrium density f_t, provided that the specific relative entropy of f_t with respect to the local equilibrium is smaller. This fixes the typical size of the relative entropies.

Denote by ψ^ε_t the density with respect to μ^0 of the local equilibrium at time t in $\Lambda = [-\varepsilon^{-1}, \varepsilon^{-1}]^d \cap \mathbb{Z}^d$ with periodic boundary conditions: fixed $\lambda(z,t)$ on $\mathbb{T}^d_1 \times \mathbf{R}_+$,we set

$$\psi^\varepsilon_t(\eta) = Z^{-1}_\varepsilon \prod_{x\in\Lambda} \exp\Big\{\sum_{\beta=0}^{d+1} \varepsilon\lambda_\beta(\varepsilon x, t) I_\beta(\eta_x)\Big\}. \qquad (6.29)$$

The specific relative entropy, in terms of the densities, is given by

$$s(f_t|\psi_t) = \varepsilon^d \int f_t \log\frac{f_t}{\psi_t} d\mu^0$$

We choose the parameters n of the local equilibrium (6.29) in the following way. We write the expectation of the conserved quantities with respect ψ_t^ε as

$$E^{\psi_t^\varepsilon}[I_0] = \rho^{(0)} + \varepsilon\rho(\varepsilon x, t) + o(\varepsilon)$$

$$E^{\psi_t^\varepsilon}[I_\beta] = \varepsilon u_\beta(\varepsilon x, t) + o(\varepsilon), \quad \beta = 1, \ldots, d$$

$$E^{\psi_t^\varepsilon}[I_{d+1}] = \mathcal{E}^{(0)} + \varepsilon\mathcal{E}(\varepsilon x, t) + o(\varepsilon)$$

We require that $m(z,t) = (\rho(z,t), u(z,t), \mathcal{E}(z,t))$ is a smooth solution of the equations (5.20) and satisfy the Boussinesq condition (5.22).

In [2] it is proven

Theorem 6.10 *If $d \geq 3$, then the specific relative entropy $s(f_t|\psi_t^\varepsilon)$ satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} s\big(f_t|\psi_t^\varepsilon\big) = 0 \tag{6.30}$$

The relative entropy method requires the smoothness of the solution of the hydrodynamic equation, which, in the case of the incompressible Navier-Stokes equations, is assured to exist at least locally in time for general initial data and globally in time for a suitable class of initial data [29].

Theorem 6.10 correctly pins down the chemical potentials λ and, together with the entropy inequality, implies a law of large numbers for the empirical measures (5.24) as stated in Theorem 5.1. In fact, for any positive a and any smooth function $J(z)$ we denote by A the event

$$\{\eta \in \{0,1\}^{\mathcal{Z}^d} \; : | \int J(z)\nu_\beta^\varepsilon(dz,t) - \int dz J(z) m_\beta(z,t)| > a\}$$

where m_β is the solution of the Navier-Stokes equations and the Boussinesq condition. One can prove the following estimate for the probability of the event A in the local equilibrium:

$$P^{\psi_t^\varepsilon}(A) \leq \text{const.} \exp[-\text{const.}\varepsilon^{2-d}\}.$$

Hence the convergence follows from Theorem 6.10 and the entropy inequality. Note that this is one of the points in which the condition $d > 2$ is essential.

Since we look at times of order ε^{-2}, we expect that the local equilibrium ψ_t^ε, which works fine on the Euler scale, will be not sufficient for our purposes and in fact to actually prove Theorem 6.9 we need to modify the local equilibrium by adding a correction of order ε^2 to ψ_t^ε. This correction is not in a product form. The specific structure of the correction is dictated by the non gradient method.

The local equilibrium ψ_t^ε is replaced by the density $\tilde{\psi}_t$ defined as follows:

$$\tilde{\psi}_t = \tilde{Z}_t^{-1} \exp\Big\{\varepsilon \sum_{x\in\Lambda}\sum_{\beta=0}^{d+1}(\omega * \lambda_\beta(\,\cdot\,,t))(\varepsilon x,t) I_\beta(\eta_x) \\ + \varepsilon^2 \Phi(\eta)\Big\}.$$

Here the convolution is

$$[\omega * \lambda(\,\cdot\,,t)](x) = \sum_{y\in\Lambda} \omega(x-y)\lambda(\varepsilon y,t),$$

and

$$\Phi(\eta) = -\sum_{x\in\Lambda}\sum_{\alpha=1}^{d}\sum_{\beta=0}^{d+1} \partial_\alpha \lambda_\beta(\varepsilon x,t)(\omega * \tau_y F_\alpha^\beta(\delta))_x \tag{6.31}$$

is the correction of order ε^2 discussed above. $F_\alpha^\beta(\delta))$ is a sequence of functions, labeled by δ, in the space $\tilde{\mathcal{G}}$ which is defined as the space $\mathcal{G}$ with the difference that in (6.1) and (6.2) the measure used to define the scalar product $< \cdot, \cdot >_0$ is the measure μ^0, corresponding to the global equilibrium. This is peculiar of the incompressible limit: since ψ_t^ε is a small perturbation of the global equilibrium it is enough to compute the diffusion coefficient by using the average with respect the global equilibrium measure μ^0. As a consequence, D will be constant in space and time. It is essential that $F_\alpha^\beta(\delta))$ are chosen in such a way that for any $\delta > 0$

$$||\sigma_i^\beta - \sum_{j=1}^{d}\sum_{\alpha=0}^{d+1} D_{i,j}^{\alpha,\beta}\nabla_j I_b - \mathcal{L}F_\alpha^\beta(\delta))||_{-1} < \delta \tag{6.32}$$

Hence, we have to find the components of the currents on the gradient space and then solve the resolvent equation to find the functions $F_\alpha^\beta(\delta)$ we need

to construct $\tilde{\psi}_t$. We notice that $\Phi(\eta)$ is of the same form of the correction to the local equilibrium in the heuristic argument for Hamiltonian particle systems in Section 3.
In the definition of $\tilde{\psi}_t$ we have also smeared $\lambda(z)$ convolving it with ω, which is the normalized characteristic function of a cubic region Λ_k of size $2k+1$, with $k = [\ell\varepsilon^{-2/d}]$, $[h]$ being the integer part of the real number h and $\ell > 0$ integer.

We show that, if the dimension $d \geq 3$, with the appropriate choice of the chemical potentials $\lambda(z,t)$, the non-equilibrium density f_t is near to the density $\widetilde{\psi}_t$ in the sense of the relative entropy, namely

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} s\big(f_t | \tilde{\psi}_t\big) = 0$$

and it is easy to show that this result implies Theorem 6.9.

The first step in the proof is to get an integral inequality for the relative entropy. The main quantity to be estimated is the derivative of the specific relative entropy called entropy production. It results

$$\frac{d}{dt} \varepsilon^{-2} s\big(f_t | \tilde{\psi}_t\big) \leq \mathcal{I},$$

with $\mathcal{I}$ the following integral

$$\mathcal{I} := \varepsilon^{d-2} \int f_t \Big(\varepsilon^{-2} \mathcal{L}^* - \frac{\partial}{\partial t}\Big) \log \widetilde{\psi}_t \, d\mu_{L,r,\theta}.$$

$\mathcal{I}$ can be computed rather explicitly and depends on the currents on the blocks Λ_k and on the derivatives of the parameters λ as well. We know how to estimate the terms arising from such a calculation, by using the variance introduced in the previous section, after having subtracted their expectations with respect the canonical measure in each block. The expectations in the limit $\varepsilon \to 0$, which corresponds, in each fixed block, to the thermodynamical limit at the equilibrium, can be computed by the equivalence of the ensembles as the expectations with respect the grand canonical measure and will give rise to the non linear terms in the limiting equations. The "subtracted currents" in the limit become the currents σ which can be treated by the non gradient arguments.

The procedure to estimate $\mathcal{I}$ consists of several steps which we omit, referring to [20] and [2] for details. The final estimate for $\mathcal{I}$ is

$$\lim_{\ell,\varepsilon} \mathcal{I} \leq \frac{c}{\gamma} \sum_{\alpha=1}^{3} \sum_{\beta=0}^{4} V\Big(\sigma_i^\beta - D_{i,j}^{\alpha,\beta} \nabla_j I_b - \mathcal{L} F_\alpha^\beta(\delta)\Big)$$
$$+ \, const.\gamma + \lim_{\ell,\varepsilon}\Big(\varepsilon^{-2} s(f_t|\tilde{\psi}_t)$$
$$+ \, \varepsilon \log E^{\psi_t^\varepsilon}\Big[\exp\Big\{\sum_x \big(\tilde{\Gamma}_x(\overline{I}_{\Lambda_k}(x))\big)\Big\}\Big]\Big) + o(1) \tag{6.33}$$

for any positive γ and for any δ, where $\lim_{\ell,\varepsilon}$ stands for $\limsup_{\ell\to\infty} \liminf_{\varepsilon\to 0}$. $\tilde{\Gamma}_x(Y)$ is an explicit function with mean zero with respect $\tilde{\psi}_t^\varepsilon$ and depends also on the parameters of the local equilibrium measure. Since it depends on the averages on big blocks, its contribution is small provided it depends quadratically on the Y's. At this point there is still the freedom to choose the chemical potentials $\lambda(z,t)$ and we can choose them in such a way that the first derivatives of $\tilde{\Gamma}$ vanish when $Y = m$. This condition is nothing but the hydrodynamical equations (5.20). This fixes the choice of the chemical potentials and singles out the hydrodynamical equations as the conditions to be imposed to the local equilibrium to control the relative entropy production.

Summarizing, the time derivative of the relative entropy is bounded in terms of two contributions of different origin:

1) a term involving the variance V of the quantity $\sigma_i^\beta + D_{i,j}^{\alpha,\beta}\nabla_j I_b - \mathcal{L}F_\alpha^\beta(\delta)$ and hence vanishing in the limit $\delta \to \infty$ since we have chosen the functions $F_\alpha^\beta(\delta)$ so that (6.32) is fulfilled;

2) a large deviation term, the third term in (6.33).

The latter involves a quadratic variation of the average of I_α over a large block Λ similar to a large deviation functional. By using properties of the local equilibrium $\tilde{\psi}_t$ it is possible to show that this term vanishes in the limit $\ell \to \infty, \varepsilon \to 0$ in the right order. We stress the fact that $\tilde{\Gamma}_x(Y)$ is quadratic in Y is crucial to prove this large deviation estimate and $\tilde{\Gamma}_x(Y)$ is quadratic if the parameters of the local equilibrium are such that the conjugate hydrodynamical fields are (smooth) solutions of the Navier-Stokes equations.

Finally, for any positive γ

$$\lim_{\delta,\ell,\varepsilon} \varepsilon^{-2} s(f_T \mid \tilde{\Psi}_T) \leq$$

$$\lim_{\delta,\ell,\varepsilon} \int_0^T \left[(\text{const.}\varepsilon^{-2} s(f_t \mid \tilde{\Psi}_t) dt + \text{const.}\gamma) \right]$$

where $\lim_{\delta,\ell,\varepsilon}$ stands for $\limsup_{\delta\to\infty} \limsup_{\ell\to\infty} \liminf_{\varepsilon\to 0}$. Theorem 6.9 is then proven by using the Gronwall Lemma.

We conclude with a technical remark. Usually the reduction of the problem from Λ to the blocks Λ_k is made by using blocks Λ_k of size of k big but independent of ε. In the limit $\varepsilon \to 0$ the volume of each block stays fixed and then k is sent to infinity so that the blocks are infinitesimal on macroscopic scale but microscopically infinite. Due to the diffusive scaling, we have to dominate an extra factor ε^{-1}, so we use the large deviation estimate in a very big block Λ_k of linear size $k = \varepsilon^{-2/d}$. This is still macroscopically infinitesimal, but much bigger than the usual blocks. Such a choice of k makes more difficult to deal with the non-gradient contribution. The dependence of k on ε forces to use a multiscale arguments and log Sobolev inequality [45] to reduce the estimate to a block of fixed size $\bar{\ell}$, sufficiently large. More details on this would be beyond the purposes of this paper, so we refer the reader to [18] and [30] for a full explanation of this matter.

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[1] Bardos, C., Golse, F., and Levermore, D. 1989, Compte Rendus del l'Acad. Sci, Ser. 1, **11**, 727. See also 1991, Jour. Stat. Phys., **63**, 323.

[2] Benois O., Esposito R. and R. Marra: 1999, J. Stat Phys. **96**, 653.

[3] Boldrighini, C., Dobrushin,R. L., and Suhov Y. M. 1983 J. Stat. Phys. **31**, 577.

[4] Boltzmann, L., 1964, Lectures on Gas Theory (Transl. from German) University of California Press, Berkeley.

[5] Boussinesq, J., 1903 Theorie analytique de la chaleur, Gauthier-Villars, Paris.

[6] Caflisch, ., 1980, Comm. Pure and Appl. Math., **33**, 651.

[7] Cercignani, C., 1988, The Boltzmann equation and its applications, Springer, New York.

[8] Cercignani, C., Illner, R., and Pulvirenti, M. 1994, The Mathematical Theory of Dilute Gases, Springer-Verlag, New York.

[9] De Masi, A., Esposito, R. , and Lebowitz, J. L., 1989, Commun. Pure and Appl. Math., **42**, 1189.

[10] De Masi, A., Ianiro, N., Pellegrinotti, S., and Presutti, E., 1984. In: Nonequilibrium Phenomena. II From Stochastics to hydrodynamics, ed. by J. L. Lebowitz, E. W. Montroll, 123. North-Holland, Amsterdam.

[11] De Masi, A., and Presutti, E., 1991, Mathematical Methods for Hydrodynamic Limits, Lecture Notes in Math 1501, Springer-Verlag, New York.

[12] De Masi, A., Presutti, E., and Scacciatelli, E., 1989, Ann. Instit. H. Poincaré A, **25**, 1.

[13] Di Perna, R. J., and Lions, P. L., 1989, Annals of Mathematics, **130**, 321.

[14] Esposito, R., Lebowitz, J. L., and Marra, R., 1994, Commun. Math. Phys. **160**, 49.

[15] Esposito, R., Lebowitz, J. L., and Marra, R., 1995, Jour. Stat. Phys. **78**, 389.

[16] Esposito, R., Lebowitz, J. L., and Marra, R., 1998, Jour. Stat. Phys., **90** 1129.

[17] Esposito, R., and Marra, R., 1993, J. Stat. Phys., **74**, 981.

[18] Esposito, R., and Marra, R., and Yau, H.T., 1994, Review in Math. Phys. **6**, 1233.

[19] Esposito, R., and Marra, R., and Yau, H.T., 1994, Nato ASI Series B: Physics **324**, 43. *On three levels: Micro-Meso and Macro- Approaches in Physics*, M. Fannes, C. Maes, A. Verbeure edt.s

[20] Esposito, R., and Marra, R., and Yau, H.T., 1996, Commun. Math. Phys. **182**, 395.

[21] Friedman, A., 1964, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, N. J. .

[22] Frisch,U., d'Humieres, D., Hasslacher, B., Lallemand, P., Pomeau, and Rivet, J. P. 1987, Complex Systems **1**, 649.

[23] Guo, M., Papanicolau, G., and Varadhan, S. R. S. 1988 Comm. Math. Phys. **118**, 31.

[24] Grad, H. 1963, in Rarified Gas Dynamics, J. A. Laurmann, ed., Vol I, 26.

[25] Hilbert, D. 1953, Grundzügeiner allgemeinen Theorie der linearen Integralgleichungen, Chelsea, New York.

[26] Kipnis, C., and Landim, C. 1999, Scaling Limits for Interacting Particle Systems, Springer-Verlag, Heidelberg.

[27] Kipnis, C., Olla, S., and Varadhan ,R. S. R. 1989, Commun. Pure Appl. Math,**42**, 115.

[28] Lachowicz, M. 198,7 Math. Meth. Appl. Sci., **93**, 27.

[29] Ladyzhenskaya, O.A. 1969, The Mathematical Theory of Viscous Incompressible Flows, Gordon and Breach, New York.

[30] Landim, C., Olla, S. and Yau, H. T. 1997 Comm. Pure Appl. Math. **L**, 149.

[31] Landim, C., Olla, S. and Yau, H. T. 1996, Annals of Probability **24**, 1779.

[32] Landim, C., and Yau, H. T. 1997, Probab. Theory Relat. Fields **108**, 321.

[33] Lanford, O. E. 1973, J. Moser ed., 1. Lecture Notes in Physics **38**, Springer, Berlin, Heidelberg.

[34] Liverani C. and Olla, S. 1996, Probab. Theory and Rel. Fields **109**, 401.

[35] Nishida, 1978 Commun. Math. Phys.**61**, 119.

[36] Olla, S., Varadhan R. S. R. and Yau, H. T. 1993, Comm. Math. Phys. **155**, 523.

[37] Quastel, J. and Yau ,H.T. 1998 Ann. Math. 148 , 51.

[38] Rezakhanlou, F. 1991, Commun. Math. Phys., **140**, 417.

[39] Spohn, H. 1991, Large Scale Dynamics of Interacting Particles, Springer-Verlag, New York.

[40] Spohn, H. 1982, Ann. Phys. **141**, 353.

[41] Sethuraman, . Varadhan, S.R.S. and Yau, H.T. 1999, preprint.

[42] Ukai, S., and Asano, K. 1983, Hokkaido Math. Jour., **12**, 303.

[43] Varadhan, S. R. S. 1994, in Asymptotic Problems in Probability Theory: Stochastic Models and Diffusion on Fractals, K.D. Elworthy and N.

Ikeda eds, Pitman Research Notes in Math., **283**, 75–128 J. Wiley & Sons, New York.

[VY] Varadhan, S. R. S., and Yau, H. T. 1997, Diffusive Limit of Lattice Gas with Mixing Conditions. Asian J. Math. **1**, 623.

[44] Yau, H. T. 1991, Letters Math. Phys. **22**, 63.

[45] Yau H. T. 1997, Probab. Th. Rel. Fields **109**, 507.