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A KINETIC MODEL OF INTERFACE MOTION

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Dedicated to Francesco Guerra on his sixtieth birthday

We study a kinetic model for a system of two species of particles interacting via a repulsive long range potential and with a reservoir at fixed temperature. The interaction between the particles is modeled by a Vlasov term and the thermal bath by a Fokker–Planck term. We show that in the diffusive and sharp interface limit the motion of the interfaces at low temperature is described by a Stefan problem or a Mullins-Sekerka motion, depending on the time scale.

Keywords: Segregation; interface motion; sharp interface limit; kinetic models.

1. Introduction

Systems of particles interacting via a weak long range potential on the lattice have been introduced in a series of papers[1–3] to study segregation phenomena and their behavior has been widely investigated. The macroscopic evolution of the conserved order parameter is ruled by a nonlinear nonlocal integral differential equation having nonhomogeneous stationary solutions at low temperature, corresponding to the presence of two different phases separated by interfaces. When the phase domains are very large compared to the size of the interfacial region (so-called sharp interface limit) the interface motion is described in terms of a Stefan-like problem or the Mullins–Sekerka motion depending on the time scale.[2]

It is well known that systems of interacting particles in the real space (as opposite to the lattice) are much more difficult to study. We propose to start this analysis by considering kinetic models of particles. We study in this paper a kinetic model of a binary mixture of particles interacting via a weak long range potential and in contact with a reservoir at fixed temperature. We show that this model undergoes phase segregation and that in the sharp interface limit the evolution is again given in terms of a Stefan-like problem or the Mullins–Sekerka model.

The equations for the one-particle distributions $f_i(x,v,\tau)$ are

$$\partial_\tau f_i + v\cdot\nabla_x f_i + F_i\cdot\nabla_v f_i = L_\beta f_i \quad i=1,2\,, \quad i\neq j\,, \tag{1.1}$$

where β is the inverse temperature of the heat reservoir modeled by the Fokker–Planck operator on the velocity space $\mathbb{R}^3$

$$L_\beta f_i := \nabla_v\cdot\left(M_\beta\nabla_v\left(\frac{f_i}{M_\beta}\right)\right), \qquad M_\beta(v)=\left(\frac{2\pi}{\beta}\right)^{-3/2}\exp\left(-\frac{\beta|v|^2}{2}\right)$$

and F_i are the self-consistent forces, whose potential has inverse range γ, representing the repulsion between particles of different species:

$$F_i(x,\tau) = -\nabla_x\int dx'\gamma^3 U(\gamma|x-x'|)\int dv f_j(x',v,\tau)\,, \quad i=1,2\,, \quad i\neq j\,.$$

Our system is contained in a 3-dimensional torus (to avoid boundary effects) and $U(r)$ is a non-negative, smooth function on $\mathbb{R}_+$ with compact support. This evolution conserves the densities of the two species. Beyond the spatially constant equilibria, there may be other spatially nonhomogeneous stationary solutions. Indeed, general entropy arguments show that the stationary solutions of these equations are local Maxwellians with mean value $u=0$, variance $T=\beta^{-1}$, and densities $\rho_i=\int dv f_i(x,v,\tau)$ satisfying

$$T\log\rho_i(x) + \int dx'\gamma^3 U(\gamma|x-x'|)\rho_j(x') = C_i\,, \quad i=1,2\,, \quad i\neq j\,. \tag{1.2}$$

Moreover, it is proved in Ref. 4, under the assumption of a monotone potential, that at low temperature there are nonhomogeneous solutions to Eq. (1.2), stable in the sense that they minimize the macroscopic free energy functional

$$\begin{aligned}\mathcal{F}(\rho_1,\rho_2) = T\int[(\rho_1\ln\rho_1)(x) + (\rho_2\ln\rho_2)(x)]dx \\ + \int U(x-y)\rho_1(x)\rho_2(y)dx\,dy\,.\end{aligned} \tag{1.3}$$

For example, in $d=1$, these solutions are called fronts and have monotonicity properties. The asymptotic values at $\pm\infty$ are the values $\bar\rho_i^{\pm}$ of the densities corresponding at equilibrium to two coexisting different phases, one reach in species 1 and the other reach in species 2. Since these solutions are unique up to a translation we fix a solution by imposing that $\rho_1(0)=\rho_2(0)$.

A different kinetic model with conservation also of momentum and energy, to take into account effects of variations of temperature and hydrodynamical flows, has been investigated in recent papers.[5,6] The interface motion in this case is driven by the hydrodynamic velocity field.

The macroscopic equations for this model are obtained in the diffusive limit: they describe the behaviour of the system on lenght scales of order ε^{-1} and time scales of order ε^{-2} in the limit of vanishing ε, where ε is the ratio between the kinetic and the macroscopic scale. Moreover, we choose $\gamma=\varepsilon$ so that the range of the potential is finite on the macroscopic scale. It can be proved that in this limit the equations become a set of two coupled parabolic equations for the densities $\rho_i(x,\tau)$

$$\beta^2 \partial_\tau \rho_i = \Delta\rho_i + \beta\nabla(\rho_i \nabla U \star \rho_j)\,, \quad i = 1,2\,, \quad i \neq j \tag{1.4}$$

where $(U \star g)(x,\tau) = \int dy U(x-y)g(y,\tau)$. These equations can be rewritten in the form of a gradient flux for the free energy functional $\mathcal{F}$

$$\partial_\tau \bar{\rho} = \nabla \cdot \left(\mathcal{M}\nabla \frac{\delta\mathcal{F}}{\delta\bar{\rho}}\right), \qquad \mathcal{M} = \beta^{-1}\begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}.$$

where $\bar{\rho} = (\rho_1, \rho_2)$, $\frac{\delta\mathcal{F}}{\delta\rho_i}$ denotes the functional derivative of $\mathcal{F}$ with respect to ρ_i and $\mathcal{M}$ is the 2×2 mobility matrix. This form of the equation is very important to study the stability properties of the stationary solutions. Since we know that the stationary solutions are minimizers of the functional $\mathcal{F}$, we expect to be able to prove that the system relaxes to that stationary state asymptotically in time, for example using the approach developed in Ref. 1 for a nonconservative equation.

In this paper we study explicitly the sharp interface limit for our model. First, we proceed in the usual way by investigating the behavior of the macroscopic equations (1.4) in the limit $L \to \infty$ where L is the typical size of the domain. The time has to be scaled as L^q as well. Using formal matched asymptotic expansions we show that for $q = 2$ the limiting equation is a nonlinear diffusion equation with Dirichlet boundary conditions on the interface. For $q = 3$ the motion of the interface is given in terms of a quasi static free-boundary problem and depends on both the mean curvature of the interface and the surface tension.

Next, we perform the macroscopic limit at the same time of the interface limit, namely we start from the kinetic model (1.2) and scale space as ϵ^{-1} and time as ϵ^{-q}, $q = 2,3$, by keeping fixed γ, and let ε go to 0. Surprisingly enough, in this limit we get the same equations as before. We like to stress that this second kind of scaling limit does not make sense for lattice models.

2. Sharp Interface Limit

In this section we start from the macroscopic equations (1.4) and investigate the limit in which the linear dimension L of the domain goes to infinity, which is the same as sending to zero as $\varepsilon = L^{-1}$ the width of the interface (sharp interface limit). We will consider the evolution equation (1.4) with $x \in \varepsilon^{-1}\Omega$, where Ω is a 3-dimensional torus, and on a time scale of order ε^{-q}, $q = 2,3$. Since we want to study the motion of an interface separating domains of the two different phases, we consider an initial condition for Eq. (1.4) in which an interface is present. The initial datum is chosen to be very close to a profile such that in the bulk its values are $\bar{\rho}_i^{\pm}$ and the interpolation between them on the interface is realized along the normal direction in each point by the fronts. Consider a smooth surface $\Gamma_0 \subset \Omega$. Let $\phi(r,\Gamma_0)$ be the signed distance of the point $r \in \Omega$ from the interface. Consider an initial profile for the densities ρ_i of the following type: at distance greater than

$O(\varepsilon)$ from the interface (in the bulk) the density profiles $\rho_i^\varepsilon(r)$ are almost constant equal to $\bar{\rho}_i^\pm$; at distance $O(\varepsilon)$ (near the interface) we choose

$$\rho_i^\varepsilon(r) = w_i(\varepsilon^{-1}\phi(r,\Gamma_0)) + O(\varepsilon) \tag{2.1}$$

where $w_i(z)$ are the fronts solutions of Eq. (1.2) in $d = 1$ and $\gamma = 1$, with asymptotic values $\bar{\rho}_i^\pm$. We remark that these fronts are also stationary solutions of (1.4). The rescaled densities $\rho_i^{\varepsilon,q}(r,t) = \rho_i(\varepsilon^{-1}r, \varepsilon^{-q}t)$ are solutions of

$$\begin{aligned} \varepsilon^{q-2}\partial_t \rho_i^{\varepsilon,q} &= \nabla_r \cdot \left(\rho_i^{\varepsilon,q} T \nabla_r \mu_i^{\varepsilon,q}\right), \\ \mu_i^{\varepsilon,q}(r,t) &= T \ln \rho_i^{\varepsilon,q}(r,t) + U_\varepsilon * \rho_j^{\varepsilon,q}(r,t), \\ U_\varepsilon(|r-r'|) &= \varepsilon^{-3} U(\varepsilon^{-1}|r-r'|). \end{aligned} \tag{2.2}$$

We look for a solution to Eq. (2.2) in the form of a Taylor series in ε. The presence of the interface forces us to use two different expansions in the bulk (outer expansion) and close to the interface (inner expansion). Far from Γ_t we write

$$\rho_i^{\varepsilon,q}(r,t) = \rho_i^{(0,q)}(r,t) + \varepsilon \rho_i^{(1,q)}(r,t) + \cdots \tag{2.3}$$

while near the interface

$$\rho_i^{\varepsilon,q}(r,t) = \tilde{\rho}_i^{\varepsilon,q} = \tilde{\rho}_i^{(0,q)}(z,r,t) + \varepsilon \tilde{\rho}_i^{(1,q)}(z,r,t) + \cdots. \tag{2.4}$$

The functions $\tilde{\rho}_i^{(i)}(z,r,t)$ depend also on a fast variable z, defined as $z = \varepsilon^{-1}\phi(r,\Gamma_t)$, where $\Gamma_t \subset \Omega$ is the interface at time t. The variable z takes into account the orthogonal displacement from the interface and has to be thought as a function of r and t. The outer normal $\nu(r,t)$ to Γ_τ in the point $x(r)$ where the perpendicular through r intersects the interface is the gradient of ϕ and the velocity $V(r,t)$ of the interface is given by $V = \partial_t \phi$.

The matching between the outer and inner expansion takes place in a neighborhood of the interface at a distance that goes to zero when ε tends to zero. We require that

$$\rho_i^{\varepsilon,q}(x(r) + \varepsilon z \nu(r), t) \approx \tilde{\rho}_i^{\varepsilon,q}(z,r,t)$$

for $z = \varepsilon^{-1+a}$, $a \in (0,1)$. Moreover, we require that $\tilde{\rho}_i^{\varepsilon,q}$ satisfy at any order the following front centering condition:

$$\frac{d}{dh}\int_{-\infty}^{+\infty} dz |\tilde{\rho}_i^{\varepsilon,q}(z,r,t) - w_i(z+h)|^2 = 0. \tag{2.5}$$

We plug the ε-series in Eq. (2.2) and equate order by order, by taking into account the matching conditions. We do not give any detail in this section for lack of space and simply state the result. We will explain how the method works starting from the kinetic model in the next section. The limiting equations are:

Case q = 2

$$\begin{cases} \partial_t \rho_i^{(0)} = \nabla_r \cdot \left(T\rho_i^{(0)} \nabla_r \mu_i^{(0)} \right) & \text{for } r \in \Omega \setminus \Gamma_t \\ \lim\limits_{h\to 0^\pm} \rho_i^{(0)}(r+h\nu, t) = \bar{\rho}_i^\pm & \text{for } r \in \Gamma_t \\ \rho_i^{(0)}(r,0) = \hat{\rho}_i(r) & \text{for all } r \text{ in } \Omega \end{cases} \tag{2.6}$$

where $\mu_i^{(0)} = T\ln\rho_i^{(0)} + \rho_i^{(0)} \int drU(r)$, $\bar{\rho}_i^\pm$ are the asymptotic values of the fronts w_i and $\hat{\rho}_i$ is a suitable initial datum. The interface moves with velocity given by

$$V = \frac{\left[T\rho_i^{(0)} \nu \cdot \nabla_r \mu_i^{(0)} \right]_-^+}{[w_i]_{-\infty}^{+\infty}} \tag{2.7}$$

where the symbol $[g]_-^+$ denotes the difference $\lim_{h\to 0^+} g(r+\nu h) - \lim_{h\to 0^-} g(r+\nu h)$ while $[w_i]_{-\infty}^{+\infty}$ is the difference between the asymptotic values $\bar{\rho}_i^\pm$. Though V seems to depend on the index i, in virtue of the symmetry which links together $\lim_{z\to\pm\infty} w_1$ and $\lim_{z\to\pm\infty} w_2$, the velocity of the interface is well defined and indeed does not depend on the index i. The problem (2.6)–(2.7) is a free-boundary partial differential equation, sometimes called a Stefan problem. The interface moves on this scale of time because of the difference of density on the two sides of the interface. The evolution of the densities is ruled by a diffusive equation with Dirichlet boundary conditions on a boundary moving according to the densities.

Case q = 3

$$\begin{cases} \triangle_r \mu_i^{(1)}(r,t) = 0 & \text{for } r \in \Omega \setminus \Gamma_t \\ (\mu_1 - \mu_2)(r,t) = \dfrac{K(r,t)S}{\bar{\rho}_1^+ - \bar{\rho}_2^-} & \text{for } r \in \Gamma_t \end{cases} \tag{2.8}$$

and the interface moves with velocity

$$V = \frac{\left[T\rho_i^{(0)} \nu \cdot \nabla_r \mu_i^{(1)} \right]_-^+}{[w_i]_{-\infty}^{+\infty}} \tag{2.9}$$

where K is the mean curvature and S is the surface tension given by

$$\begin{gathered} \frac{1}{2} \int dzdz'(z-z') \sum_{i\neq j} [w_i'(z) \tilde{U}(z-z') w_j(z')] \,, \\ \tilde{U} = \int dy_1 dy_2 U(z, y_1, y_2) \,. \end{gathered} \tag{2.10}$$

It is possible to show that there exists a unique solution to the problem (2.8)–(2.9). The interface on the slower scale of time ε^{-3} feels the surface tension effects. We notice that this problem is similar to the so-called Mullins–Sekerka flow.

3. Sharp Interface Limit and Diffusive Limit

In this section we come back to the kinetic model, described by equations (1.1) and perform the macroscopic limit in such a way that at the same time the interface becomes sharp, namely we scale position and time as ε^{-1} and ε^{-q} respectively, while keeping fixed (equal to 1) γ. We recall that here ε has the meaning of ratio between the kinetic and macroscopic scales. The width of the interface on the macroscopic scale is then of order ε, so that in the limit $\varepsilon \to 0$ the interface becomes sharp. The rescaled density distributions $f_i^\varepsilon(r,v,t) = f_i(\varepsilon^{-1}r, v, \varepsilon^{-q}t)$, with $f_i(x,v,\tau)$ solutions of (1.1), are solutions of

$$\begin{aligned} &\varepsilon^q \partial_t f_i^\varepsilon + \varepsilon v \cdot \nabla_r f_i^\varepsilon + \varepsilon F_i^\varepsilon \cdot \nabla_v f_i^\varepsilon = L_\beta f_i^\varepsilon\,, \\ &F_i^\varepsilon(r,\tau) = -\nabla_r \int dr' \varepsilon^{-3} U(\varepsilon^{-1}|r-r'|) \int dv' f_j^\varepsilon(r',v',t) =: -\nabla_r g_i^\varepsilon\,. \end{aligned} \tag{3.1}$$

We consider a situation in which initially an interface is present, as in the previous section. Since the stationary nonhomogeneous solution of Eq. (1.1) is given by the Maxwellian multiplied by the front density profile w_i we let our system start initially close to that stationary solution and choose as initial datum $f_i^\varepsilon(r,v) = M_\beta(v)\rho_i^\varepsilon$, with the density profiles (2.1) as in the previous section. We use also in this case the method of the matching expansions and look for a solution to (3.1) in the form

$$\begin{aligned} &f_i^\varepsilon(r,v,t) = f_i^{(0)}(r,v,t) + \varepsilon f_i^{(1)}(r,v,t) + \cdots\,, \\ &f_i^\varepsilon(r,v,t) = \tilde f_i^\varepsilon(z,r,v,t) = \tilde f_i^{(0)}(z,r,v,t) + \varepsilon \tilde f_i^{(1)}(z,r,v,t) + \cdots \end{aligned}$$

where the first line is the expansion in the bulk (outer) while the second one is close to the interface (inner). The outer expansion and the inner one are matched at a distance ε^a, $a \in (0,1)$. In this way one obtains links between the values of $f_i^{(k)}$ almost next to Γ_τ and those of $\tilde f_i^{(h)}$ very far away from Γ_τ with respect to the z variable. In other words, we have the following *matching conditions*

$$\begin{aligned} f_i^{(0)}(r,v,t) &= \tilde f_i^{(0)}(z,r,v,t) && \text{(for } r \text{ next to } \Gamma_t \text{ and } z \text{ very large)}\,, \\ f_i^{(1)} + z\nu \cdot \nabla_r f_i^{(0)} &= \tilde f_i^{(1)} && (\quad\text{''}\quad)\,, \\ f_i^{(2)} + z\nu \cdot \nabla_r f_i^{(1)} + \frac{z^2}{2}(\nu\cdot\nabla_r)^2 f_i^{(0)} &= \tilde f_i^{(2)} && (\quad\text{''}\quad)\,. \\ &\vdots \end{aligned}$$

Finally, we impose the front centering condition (2.5), where now the density $\tilde\rho^{\varepsilon,q}$ is given by $\int dv \tilde f_i^\varepsilon(z,r,v,t)$.

Let us start by the case $q=2$. By inserting the outer expansion in the equation we get for the three lowest orders

$$\begin{aligned} &(\varepsilon^{-2}) \quad L_\beta f_i^{(0)} = 0\,, \\ &(\varepsilon^{-1}) \quad v \cdot \nabla_r f_i^{(0)} + F_i^{(0)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(1)}\,, \\ &(\varepsilon^{0}) \quad\ \ \partial_t f_i^{(0)} + v \cdot \nabla_r f_i^{(1)} + F_i^{(0)} \cdot \nabla_v f_i^{(1)} + F_i^{(1)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(2)} \end{aligned}$$

and we readily deduce from the first equation (order (ε^{-2})) that $f_i^{(0)}(r,v,t) = \rho_i^{(0)}(r,t)M_\beta(v)$. In view of that, the equation at order ε^{-1} is rewritten in this way:

$$M_\beta v \cdot (\nabla_r \rho_i^{(0)} + \beta\rho_i^{(0)}\nabla_r g_i^{(0)}) = L_\beta f_i^{(1)}$$

that gives

$$f_i^{(1)} = -TM_\beta v \cdot \left(\nabla_r \rho_i^{(0)} + \beta\rho_i^{(0)}\nabla_r g_i^{(0)}\right),$$

where $g_i^{(0)} = \rho_i^{(0)} \int U(r)dr$. By integrating the ε^0 order equation over the velocity all the terms vanish, but the first two on the l.h.s.:

$$\partial_t \rho_i^{(0)} - T^2\Big(\triangle_r \rho_i^{(0)} + \beta\nabla_r \cdot \left(\rho_i^{(0)}\nabla_r g_i^{(0)}\right)\Big) = 0\,.$$

This equation is exactly the first of Eq. (2.6) with $\mu_i^{(0)}$ the order zero chemical potential.

In order to go on with the inner expansion, we replace the derivative operators that act on f_i^ε with those corresponding to $\tilde{f}_i^\varepsilon$. For a function $h(r,t) = \tilde{h}(z,r,t)$ we have

$$\nabla_r h = \varepsilon^{-1}\nu\partial_z \tilde{h} + \tilde{\nabla}_r \tilde{h}\,; \quad \partial_t h = \varepsilon^{-1} V\partial_z \tilde{h} + \partial_t \tilde{h} \tag{3.2}$$

where $\tilde{\nabla}_r$ is the gradient with respect only to the r coordinate. Moreover, thanks to the fact that $\nu \cdot \tilde{\nabla}_r \tilde{w} = 0$, the following holds too

$$\triangle_r h = \varepsilon^{-2}\partial_z^2 \tilde{h} + \varepsilon^{-1}(\nabla_r \cdot \nu)\partial_z \tilde{h} + \tilde{\triangle}_r \tilde{h}\,.$$

We write down the orders ε^{-2} and ε^{-1}:

$$\begin{aligned}
(\varepsilon^{-2}) \quad & v \cdot \nu\partial_z \tilde{f}_i^{(0)} - \nu \cdot \nabla_v \tilde{f}_i^{(0)}\partial_z \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(0)}\,,\\
(\varepsilon^{-1}) \quad & V\partial_z \tilde{f}_i^{(0)} + v \cdot \nu\partial_z \tilde{f}_i^{(1)} + v \cdot \tilde{\nabla}_r \tilde{f}_i^{(0)} - \nu \cdot \nabla_v \tilde{f}_i^{(0)}\partial_z \tilde{g}_i^{(1)}\\
& \qquad - \nu \cdot \nabla_v \tilde{f}_i^{(1)}\partial_z \tilde{g}_i^{(0)} - \nabla_v \tilde{f}_i^{(0)} \cdot \tilde{\nabla}_r \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(1)}\,.
\end{aligned}$$

One can show that a solution of the first equation is necessarily of the form $\tilde{f}_i^{(0)} = \tilde{\rho}_i^{(0)} M_\beta$. By replacing this expression in the order ε^{-2}) one gets

$$v \cdot \nu M_\beta\Big(\partial_z \tilde{\rho}_i^{(0)} + \beta\rho_i^{(0)}\partial_z \tilde{g}_i^{(0)}\Big) = 0 \Longleftrightarrow \partial_z \tilde{\mu}_i^{(0)} = 0\,, \tag{3.3}$$

where $\tilde{\mu}_i^{(0)}$ is defined as $\mu_i^{(0)}$. Equation (3.3) is nothing but Eq. (1.2), the relation defining a front. The first of the matching conditions imposes that at infinity the density profiles have to be $\bar{\rho}_i^{\pm}$. Hence the only solution of Eq. (3.3) with these boundary conditions, after taking into account the front centering condition (2.5), is exactly $w_i(z,t)$. Since w_i are functions only of the variable z, we have $\tilde{\nabla}_r \tilde{\rho}_i^{(0)} = 0$ and $\tilde{\nabla}_r \tilde{g}_i^{(0)} = 0$, too, because, as we will see later, $\tilde{g}_i^{(0)} = \tilde{U} * \tilde{\rho}_j^{(0)}$. Thus in the ε^{-1} order there are some simplifications and, by integrating over v, we obtain

$$V\partial_z \tilde{\rho}_i^{(0)} + \int dvv \cdot \nu\partial_z \tilde{f}_i^{(1)} = 0\,. \tag{3.4}$$

By integrating (3.4) over z,

$$V\big[\tilde{\rho}_i^{(0)}\big]_{-\infty}^{+\infty} = \left[\int dv(\nu\cdot v)\tilde{f}_i^{(1)}(z,v,r,t)\right]_{-\infty}^{+\infty}.$$

By the second of the matching conditions for z very large we have that

$$\nu\cdot\int dvv\tilde{f}_i^{(1)} = \nu\cdot\int dvv(f_i^{(1)} + z\nu\cdot\nabla_r f_i^{(0)}) = \nu\cdot\int dvvf_i^{(1)}.$$

The second equality comes from the fact that $f_i^{(0)}$ is an even function of v. By using the explicit expression of $f_i^{(1)}$ we get that the above integral is equal to $-\frac{\rho_i^{(0)}}{\beta}\nu\cdot\nabla_r\mu_i^{(0)}$. On the other hand, the first of the matching conditions implies $\left[\tilde{\rho}_i^{(0)}\right]_{-\infty}^{+\infty} = [\bar{\rho}^+ - \bar{\rho}^-]$, so that from Eq. (3.4) the expression for the velocity of the interface is given by Eq. (2.7). This completes the analysis of the case $q=2$.

We now examine the $q=3$ case. Replacing the outer expansion in the rescaled expansion and equating order by order, one gets:

$$(\varepsilon^{-3})\qquad L_\beta f_i^{(0)} = 0\,,$$
$$(\varepsilon^{-2})\qquad v\cdot\nabla_r f_i^{(0)} + F_i^{(0)}\cdot\nabla_v f_i^{(0)} = L_\beta f_i^{(1)}\,,$$
$$(\varepsilon^{-1})\qquad v\cdot\nabla_r f_i^{(1)} + F_i^{(0)}\cdot\nabla_v f_i^{(1)} + F_i^{(1)}\cdot\nabla_v f_i^{(0)} = L_\beta f_i^{(2)}\,.$$

From the first one, we deduce that $f_i^{(0)} = \rho_i^{(0)}M_\beta$; thus, the solution of the order ε^{-2} equation can be computed:

$$f_i^{(1)} = \rho_i^{(1)}M_\beta - TM_\beta v\cdot\left(\nabla_r\rho_i^{(0)} + \beta\rho_i^{(0)}\nabla_r g_i^{(0)}\right) = \rho_i^{(1)}M_\beta - M_\beta\rho_i^{(0)}v\cdot\nabla_r\mu_i^{(0)}\,.$$

The situation is quite different from the case $q=2$ because, as we will see next, at the order zero the density is constant so that the first relevant term is $\rho_i^{(1)}$. Let us integrate the order ε^{-1} equation in the velocities

$$-\sum_k\sum_h\int dvv_k(\partial_{r_k}\rho_i^{(0)}\partial_{r_h}\mu_i^{(0)})v_h M_\beta = 0 \Longleftrightarrow -T\nabla_r\cdot(\rho_i^{(0)}\nabla_r\mu_i^{(0)}) = 0\,.$$

The choice of the initial data implies that the only solution of that equation is the constant one; consequently $\partial_t\rho_i^{(0)} = 0$ and $\nabla_r\mu_i^{(0)} = 0$. Thus $f_i^{(1)}$ becomes simply $\rho_i^{(1)}M_\beta$. As usual, $f_i^{(2)}$ is determined by the ε^{-1}) equation, which becomes by the constancy of $\rho_i^{(0)}$ and the explicit version of $F_i^{(1)}$ that we will see later,

$$M_\beta v\cdot\nabla_r\left(\rho_i^{(1)} + \beta\rho_i^{(0)}g_i^{(1)}\right) = L_\beta f_i^{(2)}\,.$$

So, discarding the term in the kernel of L_β, one has

$$f_i^{(2)} = -TM_\beta v\cdot\nabla_r\left(\rho_i^{(1)} + \beta\rho_i^{(0)}g_i^{(1)}\right). \tag{3.5}$$

We need the next ε^0 order equation:

$$(\varepsilon^0) \qquad \partial_t f_i^{(0)} + v \cdot \nabla_r f_i^{(2)} + F_i^{(0)} \cdot \nabla_v f_i^{(2)} + F_i^{(1)} \cdot \nabla_v f_i^{(1)} + F_i^{(2)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(3)} \,.$$

Integrating in v and using (3.5), we obtain

$$-T^2 \triangle_r \Big(\rho_i^{(1)} + T\rho_i^{(0)} g_i^{(1)}\Big) = -T\rho_i^{(0)} \triangle_r \Big(\rho_i^{(1)} T[\rho_i^{(0)}]^{-1} + g_i^{(1)}\Big) = 0 \Longleftrightarrow \triangle_r \mu_i^{(1)} = 0$$

where we introduced $\mu_i^{(1)} = \rho_i^{(1)}/\beta\rho_i^{(0)} + g_i^{(1)}$, the order one correction to the chemical potential. This is the first equation of the limiting problem (2.8).

Let us turn to the inner expansion.

$$(\varepsilon^{-3}) \qquad v \cdot \nu \partial_z \tilde{f}_i^{(0)} - \nu \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(0)} \,,$$

$$(\varepsilon^{-2}) \qquad v \cdot \nu \partial_z \tilde{f}_i^{(1)} + v \cdot \tilde{\nabla}_r \tilde{f}_i^{(0)} - \nu \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(1)} - \nu \cdot \nabla_v \tilde{f}_i^{(1)} \partial_z \tilde{g}_i^{(0)} - \nabla_v \tilde{f}_i^{(0)} \cdot \tilde{\nabla}_r \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(1)} \,.$$

The first equation is exactly the same as that in the case $q = 2$. We deduce that $\tilde{\mu}_i^{(0)} = \text{cost}$. In other words $\tilde{f}_i^{(0)} = \tilde{\rho}_i^{(0)} M_\beta$, where $\tilde{\rho}_i^{(0)} = w_i$. Thus $\tilde{\nabla}_r \tilde{\rho}_i^{(0)} = 0$ and $\tilde{\nabla}_r \tilde{g}_i^{(0)} = 0$. This provides some simplifications in the second equation. If we now look for a solution in the form $\tilde{f}_i^{(1)} = \tilde{\rho}_i^{(1)} M_\beta$, we are led to the following equation:

$$v \cdot \nu M_\beta \Big(\partial_z \tilde{\rho}_i^{(1)} + \beta \tilde{\rho}_i^{(0)} \partial_z \tilde{g}_i^{(1)} + \beta \tilde{\rho}_i^{(1)} \partial_z \tilde{g}_i^{(0)}\Big) = 0 \,. \tag{3.6}$$

Taking into account that $-\beta \partial_z \tilde{g}_i^{(0)} = \partial_z \ln \tilde{\rho}_i^{(0)}$, because $\partial_z \tilde{\mu}_i^{(0)} = 0$, we get

$$v \cdot \nu M_\beta \beta \tilde{\rho}_i^{(0)} \partial_z \Big(T \tilde{\rho}_i^{(1)} (\tilde{\rho}_i^{(0)})^{-1} + \tilde{g}_i^{(1)}\Big) = 0 \Longleftrightarrow \partial_z \tilde{\mu}_i^{(1)} = 0 \tag{3.7}$$

where $\tilde{\mu}_i^{(1)}$ is defined as $\mu_i^{(1)}$ above. Let us go to order ε^{-1} to investigate the expression for the velocity of the interface:

$$V \partial_z \tilde{f}_i^{(0)} + v \cdot \nu \partial_z \tilde{f}_i^{(2)} + v \cdot \tilde{\nabla}_r \tilde{f}_i^{(1)} - \nu \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(2)} - \nu \cdot \tilde{f}_i^{(1)} \partial_z \tilde{g}_i^{(1)} - \nu \cdot \nabla_v \tilde{f}_i^{(2)} \partial_z \tilde{g}_i^{(0)} - \nabla_v \tilde{f}_i^{(0)} \cdot \tilde{\nabla}_r \tilde{g}_i^{(1)} - \nabla_v \tilde{f}_i^{(1)} \cdot \tilde{\nabla}_r \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(2)} \,.$$

Integrating over v, only two terms survive, due to the presence of the velocity gradient, the constancy of $\tilde{\mu}_i^{(0)}$ and the explicit form of $\tilde{f}_i^{(1)}$. Thus we are left with

$$V \partial_z \tilde{\rho}_i^{(0)} + \int dv v \cdot \nu \partial_z \tilde{f}_i^{(2)} = 0 \,.$$

The next step consists in integrating over z, but before that we apply the matching conditions for $\tilde{f}_i^{(2)}$. One is reduced to computing the following integral:

$$\int dv v \cdot \nu \Bigg(f_i^{(2)} + z\nu \cdot \nabla_r f_i^{(1)} + \frac{1}{2} z^2 \sum_k \sum_h \nu_k \nu_h \partial_{r_k} \partial_{r_h} f_i^{(0)}\Bigg) = \int dv v \cdot \nu f_i^{(2)}$$

because we recall that $\rho_i^{(0)}$ is constant and the maxwellian is an even function. Thus, in view of (3.5), one has

$$\begin{aligned}\int dvv\cdot\nu f_i^{(2)} &= -T\int dvM_\beta\sum_k\sum_h \nu_k v_k v_h\Big(\partial_{r_h}\rho_i^{(1)}+\beta\rho_i^{(0)}\partial_{r_h}g_i^{(1)}\Big)\\ &= -T^2\nu\cdot\Big[\beta\rho_i^{(0)}\nabla_r\Big(T\rho_i^{(1)}(\rho_i^{(0)})^{-1}+g_i^{(1)}\Big)\Big]\\ &= -T\rho_i^{(0)}\nu\cdot\nabla_r\mu_i^{(1)}\,.\end{aligned}$$

As a consequence, the velocity of the interface is given by (2.9). We are left with finding the expression of the term $g_i^{(1)}$ involving the long range potential U, which is defined in the second of (3.1), in order to determine the function $\mu_i^{(1)}(r,t)$ on the interface. We start from

$$\tilde\mu_i^{(1)}=\tilde\rho_i^{(1)}/\beta\tilde\rho_i^{(0)}+\tilde g_i^{(1)}\,,\qquad \tilde g_i^{(\varepsilon)}=\int dr'\varepsilon^{-3}U(\varepsilon^{-1}|r-r'|)\rho_j^{\varepsilon}(r',t)\,.$$

It is possible to show (along the lines in [2]) that

$$\tilde\mu_i^{(1)}=\tilde\rho_i^{(1)}/\beta\tilde\rho_i^{(0)}+\tilde U*\tilde\rho_j^{(1)}+\frac{K}{2}\int dz'(z-z')\tilde U(z-z')\tilde\rho_j^{(0)}(z')\,.$$

Multiplying both side by w_i' (we remember that $\tilde\rho_j^{(0)}=w_j$), integrating over z, summing on $i=1,2$ and noticing that there is a cancellation because w_i are solutions of the front equations (1.4), we get

$$\tilde\mu_1^{(1)}[w_1]_{-\infty}^{+\infty}+\tilde\mu_2^{(1)}[w_2]_{-\infty}^{+\infty}=\sum_{i=1,2}\frac{K}{2}\int dzdz'(z-z')[w_i'(z)\tilde U(z-z')w_j(z')]\,.$$

Finally, by using the matching conditions involving $\tilde\mu^{(1)}$ and Eq. (2.10) we get the second of Eq. (2.8), so completing the investigation of the case $q=3$.

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