

# Phase segregation and interface dynamics in kinetic systems

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Received 30 March 2005, in final form 30 September 2005

Published 28 October 2005

Online at [stacks.iop.org/Non/19/115](http://stacks.iop.org/Non/19/115)

Recommended by C P Dettmann

## Abstract

We consider a kinetic model of two species of particles interacting with a reservoir at fixed temperature, described by two coupled Vlasov–Fokker–Planck equations. We prove that in the diffusive limit the evolution is described by a macroscopic equation in the form of the gradient flux of the macroscopic free energy functional. Moreover, we study the sharp interface limit and find by formal Hilbert expansions that the interface motion is given in terms of a quasi-stationary problem for the chemical potentials. The velocity of the interface is the sum of two contributions: the velocity of the Mullins–Sekerka motion for the difference of the chemical potentials and the velocity of a Hele–Shaw motion for a linear combination of the two potentials. These equations are identical to the ones in Otto and E (1997 *J. Chem. Phys.* **107** 10177–84) modelling the motion of a sharp interface for a polymer blend.

Mathematics Subject Classification: 82B24, 82C40

## 1. Introduction

When a fluid mixture is suddenly quenched from a homogeneous equilibrium state into a thermodynamically unstable state it evolves to an equilibrium state consisting of two coexisting phases, each one richer in one species. This phenomenon is called phase segregation. There are various stages during this process, starting with the formation of interfaces very diffuse, that sharpen with time and then move slowly driven by surface tension effects. The study of a fluid mixture can be done at three different levels: microscopic, mesoscopic and macroscopic. At the macroscopic level the dynamics is described by phenomenological partial differential equations

for the coarse-grained conserved quantities, an example being, in the case of alloys, the Cahn–Hilliard equation [CH]. Microscopic models can be constructed by introducing classical dynamics of particles interacting through effective potentials attractive between like particles and repulsive between unlike particles, which can be studied by means of molecular dynamics simulations. Also suitable stochastic dynamics are widely investigated [GLP]. Mesoscopic models describe the system on intermediate time and length scales in terms of one-particle distribution functions. Famous examples are provided in kinetic theory [Ce] by the Boltzmann equation, describing the behaviour of a system of colliding particles on mesoscopic scales, in the low density limit, and by the Vlasov equation describing a system of interacting particles in the mean field limit. In [BELM1] the so-called late stages of the phase segregation process have been investigated for a model of a binary fluid whose mesoscopic dynamics is ruled by two coupled Vlasov–Boltzmann equations for the one-particle distributions. The underlying microscopic model is a system of two species of particles interacting by a repulsive long range potential between different species and collisions (short range potential). In [BELM] it is shown that in a mesoscopic limit, by suitably rescaling the range of the potential while sending the density to zero, the evolution of the microscopic system of two species of particles interacting through both collisions and a long range potential is ruled by the Vlasov–Boltzmann dynamics. Since this dynamics conserves masses, momentum and energy the hydrodynamics effects in the late stages of the coarsening process become relevant and when the fluid is well segregated with sharp interfaces between different phases the interface moves in its normal direction following the incompressible velocity field solution of the Navier–Stokes equation, while the pressure satisfies Laplace’s law relating it to the surface tension and curvature [BELM1]. In this paper, we study a similar mesoscopic model replacing the Boltzmann kernel by a Fokker–Planck operator, namely the two species interact with the same reservoir at a fixed temperature instead of colliding with each other, so that only the masses are conserved. We are in a situation in which the temperature and the momentum relax to equilibrium faster than the densities, as in the polymer blends where the viscosity is very large. This model can be seen as the kinetic description of a system of particles interacting via a weak long range potential (and with a reservoir) in the real space (as opposed to the lattice) or as the mean field limit of a stochastic system of interacting particles on the continuum (Ornstein–Uhlenbeck interacting processes). The system of particles on the continuum is more difficult than the corresponding ones on the lattice because of the control of the local number of particles: the conservation law cannot prevent locally very high densities. Systems of this kind on the lattice have been introduced in a series of papers [GLP] (and references therein) to study phase separation in alloys and their behaviour has been widely investigated. The macroscopic evolution of the conserved order parameter is ruled by a nonlinear nonlocal integral differential equation having nonhomogeneous stationary solutions at low temperature, corresponding to the presence of two different phases separated by interfaces. When the phase domains are very large compared with the size of the interfacial region (so-called sharp interface limit) the interface motion is described in terms of a Stefan-like problem or the Mullins–Sekerka motion depending on the time scale [GL]. In this paper we derive rigorously macroscopic equations for the density profiles and study by formal expansions the sharp interface limit.

The equations for the one-particle distributions  $f_i(x, v, \tau)$  are

$$\partial_\tau f_i + v \cdot \nabla_x f_i + F_i \cdot \nabla_v f_i = L_\beta f_i, \quad i, j = 1, 2, \quad i \neq j, \quad (1.1)$$

where  $\beta$  is the inverse temperature of the heat reservoir modelled by the Fokker–Planck operator on the velocity space  $\mathbb{R}^3$ ,

$$L_\beta f_i := \nabla_v \cdot \left( M_\beta \nabla_v \left( \frac{f_i}{M_\beta} \right) \right), \quad M_\beta(v) = \left( \frac{2\pi}{\beta} \right)^{-3/2} \exp \left( -\frac{\beta |v|^2}{2} \right)$$

and  $F_i$  are the self-consistent forces, whose potential has inverse range  $\gamma$ , representing the repulsion between particles of different species:

$$F_i(x, \tau) = -\nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv f_j(x', v, \tau), \quad i, j = 1, 2, \quad i \neq j. \quad (1.2)$$

Our system is contained in a three-dimensional torus (to avoid boundary effects) and  $U(r)$  is a nonnegative, smooth monotone function on  $\mathbb{R}_+$  with compact support. This evolution conserves the total masses of the two species. Beyond the spatially constant equilibria, there may be other spatially nonhomogeneous stationary solutions. To characterize the stationary solutions it is useful to consider the coarse-grained functional  $\mathcal{G}$ :

$$\begin{aligned} \mathcal{G}(f_1, f_2) := & \int dx dv [(f_1 \ln f_1)(x, v) + (f_2 \ln f_2)(x, v)] + \frac{\beta}{2} \int dx dv (f_1 + f_2) v^2 \\ & + \beta \int dx dy \gamma^3 U(\gamma(x-y)) \int dv f_1(x, v) \int dv' f_2(y, v'). \end{aligned}$$

It is easy to see that  $\mathcal{G}$  is a functional decreasing in time under the Vlasov–Fokker–Planck dynamics. In fact, only the Fokker–Planck term gives a contribution different from zero to the time derivative of  $\mathcal{G}$  which satisfies

$$\frac{d}{dt} \mathcal{G} = \sum_{i=1,2} \int dx dv \nabla_v \cdot \left( M_\beta \nabla_v \frac{f_i}{M_\beta} \right) \ln \frac{f_i}{M_\beta} = - \sum_{i=1,2} \int dx dv \frac{M_\beta^2}{f_i} \left[ \nabla_v \frac{f_i}{M_\beta} \right]^2 \leq 0 \quad (1.3)$$

and it is zero if and only if  $f_i$  are Gaussian functions of the velocity. Hence, the equilibrium states are local Maxwellian with mean value  $u = 0$ , variance  $T = \beta^{-1}$  and densities  $\rho_i = \int dv f_i(x, v, \tau)$  satisfying

$$T \log \rho_i(x) + \int dx' \gamma^3 U(\gamma|x-x'|) \rho_j(x') = C_i, \quad i = 1, 2, \quad i \neq j. \quad (1.4)$$

Moreover, the functional  $\mathcal{G}$  evaluated on functions of the form  $f_i(x, v, \tau) = M_\beta(v) \rho_i(x, \tau)$  with fixed total masses  $\int dv f_i$  coincides apart from a constant with the macroscopic free energy functional

$$\mathcal{F}(\rho_1, \rho_2) = \int dx [(\rho_1 \ln \rho_1)(x) + (\rho_2 \ln \rho_2)(x)] + \beta \int dx dy \gamma^3 U(\gamma(x-y)) \rho_1(x) \rho_2(y). \quad (1.5)$$

It is proved in [CCELM], under the assumption of a monotone potential, that at a low temperature there are nonhomogeneous solutions to (1.4), stable in the sense that they minimize  $\mathcal{F}$ . On the infinite line the nonhomogeneous minimizers of the excess free energy under fixed asymptotic values are called fronts and have monotonicity properties. The asymptotic values at  $\pm\infty$  are the values of the densities of two coexisting different phases at equilibrium, one rich in species 1 and the other rich in species 2.

The macroscopic equations, which play the role of the Cahn–Hilliard equations for this model, are obtained in the diffusive limit: they describe the behaviour of the system on length scales of order  $\varepsilon^{-1}$  and time scales of order  $\varepsilon^{-2}$  in the limit of vanishing  $\varepsilon$ , where  $\varepsilon$  is the ratio between the kinetic and the macroscopic scale. Moreover, we choose  $\gamma = \varepsilon$  so that the range of the potential is finite on the macroscopic scale. We prove in sections 2–4 that in this limit solutions of (1.1) converge to solutions of the following coupled non local parabolic equations for the densities  $\rho_i(x, t)$ :

$$\beta^2 \partial_t \rho_i = \Delta \rho_i + \beta \nabla \cdot (\rho_i \nabla U \star \rho_j), \quad i, j = 1, 2, \quad i \neq j, \quad (1.6)$$

where  $(U \star g)(x, t) = \int dy U(x - y)g(y, t)$ . These equations can be rewritten in the form of a gradient flux for the free energy functional  $\mathcal{F}$  ((1.5) with  $\gamma = 1$ ):

$$\partial_t \bar{\rho} = \nabla \cdot \left( \mathcal{M} \nabla \frac{\delta \mathcal{F}}{\delta \bar{\rho}} \right), \quad \mathcal{M} = \beta^{-1} \begin{Bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{Bmatrix}, \quad (1.7)$$

where  $\bar{\rho} = (\rho_1, \rho_2)$ ,  $\delta \mathcal{F} / \delta \rho_i$  denotes the functional derivative of  $\mathcal{F}$  with respect to  $\rho_i$  and  $\mathcal{M}$  is the  $2 \times 2$  mobility matrix. This form of the equation is very important to study the stability properties of the stationary solutions. Since we know that the stationary solutions are minimizers of the functional  $\mathcal{F}$ , we expect to be able to prove that the system relaxes to that stationary state asymptotically in time, for example, using the approach developed in [DOPT] for a nonconservative equation.

To describe the late stages of the segregation process, we scale position and time as  $\varepsilon^{-1}$  and  $\varepsilon^{-3}$ , respectively, while keeping fixed (equal to 1)  $\gamma$  in the Vlasov–Fokker–Planck equations (1.1). The width of the interface on the macroscopic scale is then of order  $\varepsilon$ , so that in the limit  $\varepsilon \rightarrow 0$  the interface becomes sharp. On the same scales of space and time the motion of the interfaces for models of alloys is given by the Mullins–Sekerka model [MS], a quasi-stationary boundary problem in which the mean curvature of the interface plays a fundamental role. We find similar results in our case, but with relevant differences. We choose an initial condition with an interface  $\Gamma_0$  and profiles for the densities given by front solutions with asymptotic values  $\rho_1^\pm = \rho_2^\mp = \bar{\rho}^\pm$  corresponding to the equilibrium values in the phase transition region at temperature  $T$ . In the limit  $\varepsilon \rightarrow 0$  the difference of the first correction to the chemical potentials  $\mu_i$ ,  $\psi = \mu_1^{(1)} - \mu_2^{(1)}$ , satisfies

$$\begin{aligned} \Delta_r \psi(r, t) &= 0 & \text{for } r \in \Omega \setminus \Gamma_t, \\ \psi(r, t) &= \frac{SK(r, t)}{\bar{\rho}^+ - \bar{\rho}^-} & r \in \Gamma_t, \\ V &= \frac{T}{2[\bar{\rho}^+ - \bar{\rho}^-]} \left[ \frac{1}{\bar{\rho}} (\bar{\rho}^2 - |\bar{\varphi}|^2) [v \cdot \nabla_r \psi]_-^+ + |\bar{\varphi}| v \cdot \nabla_r \zeta \right], \end{aligned} \quad (1.8)$$

where  $[\ ]_-^+$  denotes the jump across the interface  $\Gamma_t$ ,  $\bar{\rho}$  and  $\bar{\varphi}$  are the values of total density and concentration, respectively, at equilibrium,  $K$  is the curvature in  $r$  of  $\Gamma_t$  (sum of principal curvatures) and  $S$  the surface tension for this model (see appendix B). This is similar to the Mullins–Sekerka equation but for the fact that there is an extra term determining the velocity proportional to  $v \cdot \nabla_r \zeta(r, t)$ ,  $r \in \Gamma_t$  where  $\zeta(r, t) = (\bar{\rho}_1 \mu_1^{(1)} + \bar{\rho}_2 \mu_2^{(1)})(r, t)$  is the solution of

$$\begin{aligned} \Delta_r \zeta(r, t) &= 0 & \text{for } r \in \Omega \setminus \Gamma_t, \\ [\zeta]_-^+ &= 2|\bar{\varphi}|SK(r, t) & r \in \Gamma_t, \\ 0 &= [v \cdot \nabla_r \zeta]_-^+. \end{aligned} \quad (1.9)$$

The extra term is the normal interfacial velocity in the Hele–Shaw interface motion (1.9). The equations for  $\psi$  and  $\zeta$  are identical to the ones in [OE], describing the interface motion of an incompressible fluid mixture driven by thermodynamic forces, modelling a polymer blend. A discussion on that point is in section 7. In section 6 we study the sharp interface limit by means of formal expansions of the Hilbert type.

As a last remark, we observe that the usual approach in the literature to study the sharp interface limit is to start from the macroscopic equations (e.g. Cahn–Hilliard) and send to zero the ratio between the width of the interfacial region and the linear size of the phase domains. This has been studied in [MM] for the macroscopic equations (1.6) by means of formal matching expansions.

## 2. Macroscopic limit: expansion

In this section we begin the study of the hydrodynamical limit for the Vlasov–Fokker–Planck equations (1.1). We consider the diffusive scaling in which the space is scaled as  $\varepsilon$  and time as  $\varepsilon^2$  and  $\gamma = \varepsilon$ , so that the width of the interface is of order 1 on the macroscopic scale. Define

$$f_i^\varepsilon(x, v, t) := f_i(\varepsilon^{-1}x, v, \varepsilon^{-2}t), \quad i = 1, 2, \quad x \in \mathbb{T}^d, \quad v \in \mathbb{R}^d.$$

The equation for  $f_i^\varepsilon$  is

$$\partial_t f_i^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_i^\varepsilon + \frac{1}{\varepsilon} F_i^\varepsilon \cdot \nabla_v f_i^\varepsilon = \frac{1}{\varepsilon^2} L_\beta f_i^\varepsilon. \quad (2.1)$$

Here  $F_i^\varepsilon$  is the rescaled Vlasov term (1.2) with  $\gamma = \varepsilon$ :

$$F_i^\varepsilon(x, t) = -\nabla_x \int_{\mathbb{T}^d} dx' U(|x - x'|) \int_{\mathbb{R}^d} dv f_j^\varepsilon(x', v, t) := -\nabla_x U \star \rho_j^\varepsilon.$$

We substitute in (2.1) the formal power series for  $f_i^\varepsilon$  and  $F_i^\varepsilon$ :

$$f_i^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n f_i^{(n)}, \quad F_i^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n F_i^{(n)}$$

$$F_i^{(n)} = -\nabla_x U \star \int_{\mathbb{R}^d} dv f_j^{(n)}(x', v, t).$$

We get

$$\frac{1}{\varepsilon^2} L_\beta f_i^{(0)} + \frac{1}{\varepsilon} \{L_\beta f_i^{(1)} - v \cdot \nabla_x f_i^{(0)} - F_i^{(0)} \cdot \nabla_v f_i^{(0)}\}$$

$$- \sum_{n=0}^{\infty} \varepsilon^n \left\{ \partial_t f_i^{(n)} + v \cdot \nabla_x f_i^{(n+1)} + \sum_{l, l' \geq 0: l+l'-1=n} (F_i^{(l)} \cdot \nabla_v f_i^{(l')}) - L_\beta f_i^{(n+2)} \right\} = 0.$$

At each order in  $\varepsilon$  we get an equation. We write down here explicitly the first three orders:

$$\varepsilon^{-2} - L_\beta f_i^{(0)} = 0,$$

$$\varepsilon^{-1} - v \cdot \nabla_x f_i^{(0)} + F_i^{(0)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(1)},$$

$$\varepsilon^0 - \partial_t f_i^{(0)} + v \cdot \nabla_x f_i^{(1)} + F_i^{(0)} \cdot \nabla_v f_i^{(1)} + F_i^{(1)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(2)}.$$

From  $\varepsilon^{-2}$  we deduce from the properties of  $L_\beta$  that  $f_i^{(0)}$  is the Maxwellian  $M_\beta$  multiplied by a density factor depending on  $x$  and  $t$ :

$$f_i^{(0)} = \rho_i(x, t) M_\beta. \quad (2.2)$$

Replacing this expression in the second equation (order  $\varepsilon^{-1}$ ) we get

$$M_\beta v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) = L_\beta f_i^{(1)}.$$

So a solution has to be of the form

$$f_i^{(1)} = M_\beta (A + B \cdot v),$$

where  $A$  will be fixed by the equations of the next orders and  $B$  is the vector:

$$B = -\frac{1}{\beta} (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)).$$

If we put these expressions of  $f_i^{(0)}$  and  $f_i^{(1)}$  in the  $\varepsilon^0$  equation and integrate over  $v$ , remembering that

$$\int_{\mathbb{R}^3} dv M_\beta v_i v_j = \frac{1}{\beta} \delta_{ij},$$

we find the equations for the zero order densities

$$\partial_t \rho_i - \frac{1}{\beta^2} (\Delta_x \rho_i + \beta \nabla_x \cdot (\rho_i \nabla_x (U \star \rho_j))) = 0. \quad (2.3)$$

Now our aim is to show that a solution of equation (2.1) does exist and its limit as  $\varepsilon$  goes to zero is given by (2.2), with  $\rho_i$  satisfying (2.3). We try to solve (2.1) in terms of a truncated expansion

$$f_i^\varepsilon = \sum_{n=0}^K \varepsilon^n f_i^{(n)} + \varepsilon^m R_i. \quad (2.4)$$

Replacing expression (2.4) for  $f_i^\varepsilon$  in equation (2.1) we get for any  $n$  between 0 and  $K$

$$\partial_t f_i^{(n-2)} + v \cdot \nabla_x f_i^{(n-1)} + \sum_{l,l' \geq 0: l+l'-1=n-2} \left[ -\nabla_x U \star \int_{\mathbb{R}^3} dv f_j^{(l)} \right] \cdot \nabla_v f_i^{(l')} - L_\beta f_i^{(n)} = 0, \quad (2.5)$$

$f_i^{(s)} = 0$ ,  $s \leq 0$  and for the remainder

$$\partial_t R_i + \frac{1}{\varepsilon} [v \cdot \nabla_x R_i + F_i^\varepsilon \cdot \nabla_v R_i + B_i \cdot \Gamma_i] = \frac{1}{\varepsilon^2} L_\beta R_i - \varepsilon^{K-m-1} A_i, \quad (2.6)$$

where we defined

$$B_i = \sum_{n=0}^K \varepsilon^n \nabla_v f_i^{(n)}, \quad \Gamma_i = -\nabla_x U \star \int_{\mathbb{R}^3} dv R_j,$$

$$A_i = \partial_t f_i^{(K-1)} + v \cdot \nabla_x f_i^{(K)} + \varepsilon \partial_t f_i^{(K)} + \sum_{n=K-1}^{2K-1} \varepsilon^{n-K+1} \sum_{\substack{0 \leq l, l' \leq K \\ l+l'-1=n}} F_i^{(l)} \cdot \nabla_v f_i^{(l')}.$$

We will find solutions  $f_i^{(n)}$  to equations (2.5) in the next section and we will study the equation for the remainder  $R_i$  in section 5. Here we state the results.

Denote by  $(\cdot, \cdot)_-$  the following  $L_2$  scalar product and with  $\|\cdot\|_-$  the associated norm

$$(h, g)_- := \int_{\mathbb{T}^d \times \mathbb{R}^d} dx dv M_\beta^{-1}(v) \sum_{i=1,2} [h_i(x, v) g_i(x, v)].$$

Put  $A = \{A_i\}_{i=1,2}$  and  $R = \{R_i\}_{i=1,2}$ .

**Theorem 2.1.** *Given a classical solution  $\rho_i(x, t)$  of the macroscopic equations (1.6) in the time interval  $[0, T]$ , there is a constant  $C$  depending on  $T$ , such that a unique solution to (2.6) exists and satisfies the bounds*

$$\sup_{t \in [0, T]} \|R(\cdot, t)\|_- \leq C \varepsilon^{K-1-m} \|A\|_-.$$

As a consequence, we have the following corollary.

**Corollary 2.1.** *Under the assumptions of theorem 2.1 and  $m \geq 1$ ,  $K - 1 - m \geq 0$ , there is a positive constant  $\varepsilon_0$  such that for  $\varepsilon < \varepsilon_0$  there is a smooth solution  $f_i^\varepsilon(x, v, t)$  to the rescaled Vlasov–Fokker–Planck equations (2.1) satisfying for some constant  $C$*

$$\sup_{t \in [0, T]} \|f_i^\varepsilon - M_\beta \rho_i\|_- \leq C \varepsilon.$$

### 3. Macroscopic limit: expansion terms

In this section we show existence and regularity properties of  $f_i^{(n)}$ . For simplicity, we write down the proof only for  $K = 2$ , but the argument goes on for any  $K$ . The structure of equations (2.5) is very simple: they are of the form

$$L_\beta f = h \quad (3.1)$$

with  $h$  a given function. In the Hilbert space with scalar product

$$(h, g)_M = \int_{\mathbb{R}^d} dv h(\cdot, v) g(\cdot, v) M_\beta^{-1},$$

the kernel  $\mathcal{N} = \ker(L_\beta)$  is made of constants in velocity multiplied by  $M_\beta$ . Hence this equation has a solution iff  $h$  is in the orthogonal to the kernel of  $L_\beta$ , namely, iff

$$\int dv h(v) = 0. \quad (3.2)$$

Moreover, the solution is determined but for a term in the kernel which is of the form a function of  $x, t$  times the Maxwellian. Starting from the lowest order, we will see that  $h(x, v, t) = P(x, t, v) M_\beta$  with  $P$  a polynomial of the velocity with coefficients eventually depending on  $x, t$ . Equation (3.1) can be solved uniquely in the orthogonal to the null space of  $L_\beta$ . If  $P$  is a polynomial the solution is again a polynomial of the same degree of  $P$  multiplied by the Maxwellian  $M_\beta$ . In other words, if  $M_\beta P \in \mathcal{N}^\perp$  with  $P$  a polynomial, then there exists a unique  $f \in \mathcal{N}^\perp$  such that (3.1) holds. This statement can be shown by finding explicitly solutions to the problem (3.1) for different choices of the polynomial  $P$ . We are interested in polynomials of degree up to the second.

For  $n = 0, 1$  the equations (2.5) are of the form (3.1) with  $h = 0$  and  $h = b_i v_i$ , respectively, and have already been discussed in the previous section. We recall that  $f^{(1)}$  can be found as  $M_\beta(A_i + B_i v_i)$ , with  $B_i = -(1/\beta)b_i$ .  $A_i$  would be determined by the compatibility condition at the order  $n = 3$ . Since we are truncating the expansion at  $n = 2$  we can safely choose  $A_i$  equal to zero.

Let us now deal with a polynomial of degree two:

$$P(v) = a + b_i v_i + c_{ij} v_i v_j.$$

By Gaussian integration, the condition (3.2) becomes

$$a + \frac{1}{\beta} c_{ii} = 0. \quad (3.3)$$

We look for a solution of (3.1) of the following type:

$$f(v) = M_\beta(A + B_i v_i + C_{ij} v_i v_j).$$

Plugging this ansatz in our equation we find

$$\partial_{v_k} (M_\beta \partial_{v_k} (A + B_i v_i + C_{ij} v_i v_j)) = M_\beta P(v).$$

Recall that  $\partial_{v_k} v_i = \delta_{ki}$  and  $\partial_{v_k} v_i v_j = \delta_{ki} v_j + \delta_{kj} v_i$ ; then the left-hand side (lhs) of the above equation simplifies to

$$\begin{aligned} \partial_{v_k} (M_\beta (B_k + C_{kj} v_j + C_{ik} v_i)) &= M_\beta (-\beta v_k (B_k + C_{kj} v_j + C_{ik} v_i) + \delta_{kj} C_{kj} + \delta_{ik} C_{ik}) \\ &= M_\beta (2C_{ii} - \beta B_i v_i - 2\beta C_{ij} v_i v_j) \end{aligned}$$

and identifying the coefficients of the corresponding powers of  $v_i$  one gets

$$B_i = -\frac{1}{\beta} b_i, \quad C_{ij} = -\frac{1}{2\beta} c_{ij}$$

with the relation  $a = 2C_{ii}$  which is automatically verified, thanks to the compatibility condition (3.3). The parameter  $A$  is fixed by imposing  $A + C_{ii}/\beta = 0$ . Thus

$$A = -\frac{a}{2\beta}.$$

In the context of our problem the known term is always in the form of a polynomial multiplied by a Maxwellian and the coefficients of the  $v_i$  are functions of the position. In the case where only first powers of  $v$  appear, i.e.  $P(v) = a_k^{(i)} v_k$ , the  $a_k^{(i)}$  are given by

$$a^{(i)} = \nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j),$$

where  $i, j = 1, 2$  and  $i \neq j$ . When  $P(v) = a^{(i)} + b_k^{(i)} v_k + c_{hk}^{(i)} v_h v_k$  the coefficients are the following:

$$\begin{aligned} a^{(i)} &= \partial_t \rho_i + \nabla_x (U \star \rho_j) \cdot \frac{1}{\beta} (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)), \\ b_k^{(i)} &= 0, \\ c_{hk}^{(i)} &= -\frac{1}{\beta} \partial_{x_h} (\partial_{x_k} \rho_i + \beta \rho_i \partial_{x_k} (U \star \rho_j)) - \partial_{x_h} (U \star \rho_j) (\partial_{x_k} \rho_i + \beta \rho_i \partial_{x_k} (U \star \rho_j)). \end{aligned}$$

Summing up we denote by  $f_i^{(k)}$ ,  $k = 0, 1, 2$ , the following functions of  $v$  and  $x$ :

$$\begin{aligned} f_i^{(0)} &= M_\beta \rho_i(x, t), \\ f_i^{(1)} &= -\frac{1}{\beta} M_\beta v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)), \\ f_i^{(2)} &= -\frac{1}{2\beta} M_\beta \left[ \partial_t \rho_i + \nabla_x (U \star \rho_j) \cdot \frac{1}{\beta} (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) - v \cdot \frac{1}{\beta} \nabla_x \right. \\ &\quad \left. \times (v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j))) - v \cdot \nabla_x (U \star \rho_j) v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) \right], \end{aligned}$$

where  $\rho_i$  is the solution of

$$\partial_t \rho_i - \frac{1}{\beta^2} (\Delta_x \rho_i + \beta \nabla_x \cdot (\rho_i \nabla_x (U \star \rho_j))) = 0.$$

The known term  $A_i$  appearing in the equation for the remainder becomes

$$A_i = \partial_t f_i^{(1)} + v \cdot \nabla_x f_i^{(2)} + \varepsilon \partial_t f_i^{(2)} + \sum_{n=1}^3 \varepsilon^{n-1} \sum_{0 \leq l, l' \leq 2: l+l'-1=n} F_i^{(l)} \cdot \nabla_v f_i^{(l')},$$

where we recall that

$$F_i^{(n)} = -\nabla_x \int_{\varepsilon\Omega} dx' U(|x - x'|) \int_{\mathbb{R}^3} dv' f_j^{(n)}.$$

It is easy to show that the sum over  $l, l'$  is given by

$$-\nabla_x (U \star \rho_j) \cdot \nabla_v f_i^{(2)};$$

indeed  $F_i^{(1)} = 0 = F_i^{(2)}$  because the functions  $f_i^{(1)}$  and  $f_i^{(2)}$  belong to  $\mathcal{N}^\perp$  and  $F_i^{(0)} = -\nabla_x (U \star \rho_j)$ .

In conclusion, the  $f_i^{(n)}$  are always of the form  $M_\beta$  times a polynomial in  $v$  times a function of  $x, t$  which depends on the derivatives of  $\rho_i(x, t)$  solution of the macroscopic equations. If we fix an initial datum for (1.6) in  $C^2(\mathbb{T}^d)$  then the corresponding unique solution will be classical as shown in section 5 and  $f^{(n)} = \{f^{(n)}\}_i$  as well as  $A_i$  will satisfy the regularity properties

$$\|f^{(n)}\|_- \leq C, \quad \|A\|_- \leq C.$$

#### 4. Macroscopic limit: remainder

In this section we will find a solution to equation (2.6), which is a weakly nonlinear equation if  $m \geq 1$ ,  $K - 1 - m \geq 0$ , by considering first the linear problem with a given force  $F_i^\varepsilon$ . General results will grant the existence in some suitable space for this linear problem. Then, a fixed point argument applies by using  $\varepsilon$  as a small parameter. From here on we will simplify the notation by setting  $M = M_\beta$ .

Define  $\tilde{f} = f/M$  and

$$\tilde{L}_\beta \tilde{f} = \frac{1}{M} \nabla_v \cdot (M \nabla_v (\tilde{f})).$$

Moreover, we introduce the Hilbert space associated with the  $L_2$  scalar product  $(\cdot, \cdot)_M$  weighted by the Maxwellian and  $\|\cdot\|_M$  the associated norm. In this Hilbert space  $\tilde{L}_\beta$  is self-adjoint and nonpositive:

$$\begin{aligned} (g_1, \tilde{L}_\beta g_2)_M &= (\tilde{L}_\beta g_1, g_2)_M, \\ (g, \tilde{L}_\beta g)_M &= \int_{\mathbb{T}^d \times \mathbb{R}^d} dx dv M g \frac{1}{M} \nabla_v \cdot (M \nabla_v (g)) = -\|\nabla_v g\|_M^2. \end{aligned}$$

If we put  $R_i = \psi_i M$ , the equation for the remainder becomes

$$\partial_t \psi_i + \varepsilon^{-1} \left[ v \cdot \nabla_x \psi_i + \frac{F_i^\varepsilon \cdot \nabla_v (M \psi_i)}{M} + \frac{B_i \cdot \Gamma_i}{M} \right] = \varepsilon^{-2} \tilde{L}_\beta \psi_i - \varepsilon^{K-1-m} \frac{A_i}{M}. \quad (4.1)$$

In order to estimate  $\|\psi_i\|_M$  one multiplies the above equation by  $M \psi_i$  and integrates over  $x$  and  $v$ . So the first term on the lhs becomes

$$\frac{1}{2} \partial_t \|\psi_i\|_M^2,$$

while the gradient with respect to the position disappears because of the periodic boundary conditions.

We assume that the force terms  $F_i^\varepsilon$  are given functions that we will call  $\hat{F}_i$  and are such that

$$\|\hat{F}_i\|_\infty \leq \alpha_{\hat{F}};$$

hence

$$\left| \int dx dv \psi_i \hat{F}_i \cdot \nabla_v (M \psi_i) \right| = \left| \int dx dv \hat{F}_i \cdot (M^{1/2} M^{1/2} \psi_i \nabla_v \psi_i) \right| \leq \|\hat{F}_i\|_\infty \|\psi_i\|_M \|\nabla_v \psi_i\|_M,$$

where we integrated by parts ( $\hat{F}_i$  depends only on  $x$ ) and we used the Schwartz inequality. Now the term with the convolution of the remainder with the gradient of the potential is estimated in the following way:

$$\begin{aligned} & \left| \int dx dv \psi_i(x, v) \nabla_v (\rho(x) M(v)) \cdot \int dx' \nabla_x U(|x - x'|) \int dv' M(v') \psi_j(x', v') \right| \\ &= \left| \int dx dv \rho M^{1/2}(v) M^{1/2}(v) \nabla_v \psi_i(x, v) \cdot \int dx' dv' \nabla_x U M^{1/2}(v') M^{1/2}(v') \psi_j(x', v') \right| \\ &\leq |\mathbb{T}^d| \sup_{\mathbb{T}^d} |\rho| \sup_{\mathbb{T}^d} |\nabla_x U| \|\psi_j\|_M \|\nabla_v \psi_i\|_M. \end{aligned}$$

As before we first integrated by parts and then we applied the Schwartz inequality twice. Here only the lowest order in  $\varepsilon$  of the sum which constitutes  $B_i$  has been considered; the other terms are treated similarly. The last estimate is the one for  $A_i$ :

$$\int dx dv \psi_i A_i = \int dx dv M^{1/2} \psi_i \frac{A_i}{M^{1/2}} \leq \|\psi_i\|_M \|M^{-1} A_i\|_M \leq \frac{1}{2} (\|\psi_i\|_M^2 + \|M^{-1} A_i\|_M^2).$$

Summing up, we have

$$\begin{aligned} \frac{1}{2} \partial_t \|\psi_i\|_M^2 &\leq -\varepsilon^{-2} \|\nabla_v \psi_i\|_M^2 + (c_1 + \varepsilon^{-1} c_2) \|\psi_j\|_M \|\nabla_v \psi_i\|_M + \alpha_{\hat{F}_i} \varepsilon^{-1} \|\psi_i\|_M \|\nabla_v \psi_i\|_M \\ &\quad + \frac{\varepsilon^{K-1-m}}{2} (\|\psi_i\|_M^2 + \|M^{-1} A_i\|_M^2). \end{aligned}$$

Note that  $c_1$  contains powers of  $\varepsilon$  greater than  $\varepsilon^{-1}$ . Now one exploits the inequality

$$-\frac{\varepsilon^{-2} x^2}{2} + (\sigma_1 + \varepsilon^{-1} \sigma_2) x y \leq \frac{(\varepsilon \sigma_1 + \sigma_2)^2}{2} y^2, \quad (4.2)$$

but first we need to introduce the norm  $\|\psi\|_M^2 := \|\psi_1\|_M^2 + \|\psi_2\|_M^2$ , so we have

$$\begin{aligned} \frac{1}{2} \partial_t \|\psi\|_M^2 &\leq -\varepsilon^{-2} (\|\nabla_v \psi_1\|_M^2 + \|\nabla_v \psi_2\|_M^2) + (c_1 + \varepsilon^{-1} c_2) (\|\psi_2\|_M \|\nabla_v \psi_1\|_M \\ &\quad + \|\psi_1\|_M \|\nabla_v \psi_2\|_M) + \alpha_{\hat{F}} \varepsilon^{-1} (\|\psi_1\|_M \|\nabla_v \psi_1\|_M + \|\psi_2\|_M \|\nabla_v \psi_2\|_M) \\ &\quad + \frac{\varepsilon^{K-1-m}}{2} [(\|\psi_1\|_M^2 + \|\psi_2\|_M^2) + (\|M^{-1} A_1\|_M^2 + \|M^{-1} A_2\|_M^2)] \\ &\leq \frac{(\varepsilon c_1 + c_2)^2}{2} (\|\psi_1\|_M^2 + \|\psi_2\|_M^2) + \frac{\alpha_{\hat{F}}^2}{2} (\|\psi_1\|_M^2 + \|\psi_2\|_M^2) \\ &\quad + \frac{\varepsilon^{K-1-m}}{2} [(\|\psi_1\|_M^2 + \|\psi_2\|_M^2) + (\|M^{-1} A_1\|_M^2 + \|M^{-1} A_2\|_M^2)], \end{aligned}$$

where we used the inequality (4.2) in two different ways; in fact we divided the negative term into two halves and then once we chose  $c_1 = \sigma_1$  and  $c_2 = \sigma_2$  and once we put  $c_1 = 0$  and  $c_2 = \alpha_{\hat{F}}$ .

Multiplying both members by two one gets

$$\partial_t \|\psi\|_M^2 \leq \lambda \|\psi\|_M^2 + d,$$

where  $\lambda = \lambda(\alpha_{\hat{F}}) = \alpha_{\hat{F}}^2 + (\varepsilon c_1 + c_2)^2 + \varepsilon^{K-1-m}$  and  $d = \varepsilon^{K-1-m} (\|M^{-1} A_1\|_M^2 + \|M^{-1} A_2\|_M^2)$ . Integrating over time, by the Gronwall inequality

$$\begin{aligned} f(t) &\leq K(t) + \lambda \int_0^t d\tau f(\tau) \leq K(T) + \lambda \int_0^T d\tau f(\tau) \\ &\implies f(t) \leq K(T) e^{\lambda t} \leq K(T) e^{\lambda T}, \end{aligned}$$

where  $f = \|\psi\|_M^2$ ,  $K(t) = \int_0^t d\tau d(\tau)$  is a non decreasing function of time and we used the initial condition  $f(0) = 0$ .

Now consider the sequence of forces

$$\hat{F}_i^{(k)} = -\nabla_x U \star \int_{\mathbb{R}^3} dv \sum_{n=0}^K \varepsilon^n f_j^{(n)} - \varepsilon^m \nabla_x U \star \int_{\mathbb{R}^3} dv R_j^{(k-1)}$$

with  $k \geq 1$  and  $R_i^{(0)} = 0$ . Let  $\alpha_k = \max\{\|\hat{F}_1^{(k)}\|_\infty, \|\hat{F}_2^{(k)}\|_\infty\}$ , then

$$\alpha_k \leq \bar{\alpha} + \varepsilon^m C \int dx dv |R_j^{(k-1)}|,$$

where  $j$  is chosen such that it corresponds to the maximum in the definition of  $\alpha_k$  and

$$\bar{\alpha} = \sup_{x \in \mathbb{T}^d} \left| \nabla_x U \star \int dv \sum_{n=0}^K \varepsilon^n f_j^{(n)} \right|.$$

Write  $\int dv |R_j^{(k-1)}| = \int dv |M \psi_j^{(k-1)}| = \int dv M^{1/2} |M^{1/2} \psi_j^{(k-1)}|$ ; using the Schwartz inequality we get

$$\int dx dv |R_j^{(k-1)}| \leq |\mathbb{T}^d|^{1/2} \|\psi_j^{(k-1)}\|_M.$$

Thus, recalling the estimate for  $\|\psi\|_M^2$ , we can conclude that

$$\alpha_k \leq \bar{\alpha} + \varepsilon^m \mu(\alpha_{k-1}),$$

where the nondecreasing function  $\mu$  is defined by  $\mu(\alpha_k) = C(|\mathbb{T}^d|K(T) \exp(\lambda(\alpha_k)T))^{1/2}$ . By induction on  $k$  we show that  $\alpha_k \leq 2\bar{\alpha} \forall k$ . In fact

$$\alpha_1 \leq \bar{\alpha} \leq 2\bar{\alpha}.$$

Then suppose  $\alpha_{k-1} \leq 2\bar{\alpha}$ ; we have

$$\alpha_k \leq \bar{\alpha} + \varepsilon^m \mu(\alpha_{k-1}) \leq \bar{\alpha} + \varepsilon^m \mu(2\bar{\alpha}) \leq 2\bar{\alpha}$$

because we applied the inductive hypothesis, exploited the monotonicity of  $\mu$  and chose  $\varepsilon$  so small that  $\varepsilon^m \mu(2\bar{\alpha}) \leq \bar{\alpha}$ .

Denote with  $\delta\psi_i^{(k)}$  the difference  $\psi_i^{(k)} - \psi_i^{(k-1)}$ . The equation solved by  $\delta\psi_i^{(k)}$  is

$$\begin{aligned} \partial_t(\delta\psi_i^{(k)}) + \varepsilon^{-1} \left[ v \cdot \nabla_x(\delta\psi_i^{(k)}) + \frac{\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k)}) - \hat{F}_i^{(k-1)} \cdot \nabla_v(M\psi_i^{(k-1)})}{M} + \frac{B_i \cdot \Gamma_i}{M} \right] \\ = \varepsilon^{-2} \tilde{L}_\beta(\delta\psi_i^{(k)}), \end{aligned}$$

where it is understood that  $\Gamma_i$  contains  $\delta\psi_j^{(k)}$  and no more  $\psi_j$ . Summing and subtracting the quantity  $\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k-1)})$  one has

$$\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k)}) - \hat{F}_i^{(k-1)} \cdot \nabla_v(M\psi_i^{(k-1)}) = \hat{F}_i^{(k)} \cdot \nabla_v(M\delta\psi_i^{(k)}) + \delta\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k-1)}),$$

where

$$\delta\hat{F}_i^{(k)} = \hat{F}_i^{(k)} - \hat{F}_i^{(k-1)} = -\varepsilon^m \nabla_x U \star \int dv' M \delta\psi_j^{(k-1)}.$$

If one multiplies the equation for  $\delta\psi_i^{(k)}$  by  $M\delta\psi_i^{(k)}$  and integrates in space and velocities, it is possible to replicate the above estimates for the norm of the remainder. Only one thing is worth noting: the known term with  $A_i$  is now replaced by the following quantity:

$$\int dx dv \delta\psi_i^{(k)} \delta\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k-1)}) = - \int dx dv M\psi_i^{(k-1)} \delta\hat{F}_i^{(k)} \cdot \nabla_v(\delta\psi_i^{(k)})$$

which one estimates in this way:

$$\begin{aligned} \varepsilon^m \left| \int dx dv M(v)(\psi_i^{(k-1)} \nabla_v \delta\psi_i^{(k)})(x, v) \int dx' \nabla_x U(|x-x'|) \int dv' M(v') \delta\psi_j^{(k-1)}(x', v') \right| \\ \leq \varepsilon^m \sup |\nabla_x U| \left( \int dx dv |M\psi_i^{(k-1)} \nabla_v \delta\psi_i^{(k)}| \right) \left( \int dx' dv' M |\delta\psi_j^{(k-1)}| \right) \\ \leq \varepsilon^m |\mathbb{T}^d|^{1/2} \sup |\nabla_x U| \|\psi_i^{(k-1)}\|_M \|\nabla_v \delta\psi_i^{(k)}\|_M \|\delta\psi_j^{(k-1)}\|_M \\ \leq \varepsilon^m c \|\nabla_v \delta\psi_i^{(k)}\|_M \|\delta\psi_j^{(k-1)}\|_M \leq \frac{\varepsilon^m c}{2} (\|\nabla_v \delta\psi_i^{(k)}\|_M^2 + \|\delta\psi_j^{(k-1)}\|_M^2). \end{aligned}$$

In  $c$  the bound for  $\|\psi_i^{(k-1)}\|_M$  is also present. In brief we have the following situation:

$$f'_k \leq C f_k + \theta f_{k-1}$$

for some  $C$ ;  $\theta$  depends on  $\varepsilon$  and is small as we like if  $m \geq 1$  and, of course,  $f_k = \|\delta\psi^{(k)}\|_M^2$  with the same notation as above. By integrating in time and using the Gronwall inequality we obtain

$$f_k \leq \int_0^T \theta f_{k-1} e^{CT} \leq \int_0^T \theta e^{CT} \int_0^T \theta e^{CT} f_{k-2} \leq \dots \leq \text{const}(\theta e^{CT} T)^k;$$

thus, by a standard argument, we conclude that the sequence  $\{\psi_i^{(k)}\}$  is a Cauchy sequence and the limit  $\psi$  is the unique solution of (4.1) with bounded norm  $\|\psi\|_M$ .

## 5. Limiting equation

We follow a strategy similar to the one used in the previous section: we consider first a linear problem, prove existence for it and then use a fixed point argument to give the existence for the full nonlinear equation. Since we do not have at our disposal a small parameter we use compactness arguments and the Schauder fixed point theorem [E]. We seek for weak solutions in the following sense:

Let  $W$  be the Hilbert space

$$W(0, T; H^1, H^{-1}) := \left\{ f : f \in L^2(0, T; H^1), \frac{df}{dt} \in L^2(0, T; H^{-1}) \right\}.$$

Let  $H^1(\mathbb{T}^d)$  and  $H^{-1}(\mathbb{T}^d)$  be Sobolev spaces on the torus with norms

$$\begin{aligned} |v|_2^2 &= \int_{\mathbb{T}^d} |v|^2, & \|v\|_1^2 &= |v|_2^2 + |\nabla v|_2^2, \\ \|v\|_{-1} &= \sup_{u \in H^1} [2(u, v) - \|u\|_1^2] = \int dk \frac{|\hat{v}(k)|^2}{1+k^2}, \end{aligned}$$

with  $(\cdot, \cdot)$  scalar product in  $L^2$ .

$$\|v\|_W^2 = \int_0^T [\|v(t)\|_1^2 + \|v'(t)\|_{-1}^2] dt$$

with  $v' = dv/dt$ . Let  $W_1$  be the convex subset of  $W$ :

$$W_1 = \left\{ v \in W : \int_{\mathbb{T}^d} v(x, t) = 1 \text{ a.e in } [0, T] \right\}.$$

We say that  $\rho$  is a weak solution of the linear problem (5.1) below if for  $\bar{\rho} \in L^2(\mathbb{T}^d)$  and for all  $v \in H^1(\mathbb{T}^d)$  and a.a.  $0 \leq t \leq T$

$$\beta^2(v, \rho') + (\nabla v, \nabla \rho + \beta \rho \nabla U \star h) = 0$$

and  $\rho(\cdot, 0) = \bar{\rho}(\cdot)$ .

We remark that since  $\rho \in W$  implies  $\rho \in C([0, T]; L^2(\mathbb{T}^d))$  we have that  $\rho(0) \in L^2(\mathbb{T}^d)$ .

**Theorem 5.1.** *For any  $h \in L^1(\mathbb{T}^d)$  and  $\bar{u} \in L^2(\mathbb{T}^d)$  there exists a unique solution in  $W_1$  to the following Cauchy problem:*

$$\begin{aligned} \beta^2 \partial_t u &= \Delta u + \beta \nabla \cdot (u \nabla (U \star h)), \\ u(\cdot, 0) &= \bar{u}(\cdot). \end{aligned} \tag{5.1}$$

**Proof.** Since  $h \in L^1(\mathbb{T}^d)$  and  $\nabla U$  as well as  $\nabla^2 U$  are bounded we have  $\nabla(U \star h)$  and  $\nabla^2(U \star h)$  in  $L^\infty([0, T] \times \mathbb{T}^d)$ . Hence by standard arguments [E] there exists a solution in  $W$ . Since the equation is in the form of divergence, the total mass is conserved so that the solution is in  $W_1$ .

Moreover, we have some useful *a priori* estimates for the solution of (5.1) (indeed the proof of existence can be achieved by approximation methods and these *a priori* estimates). Denote by  $|u|_2$  the norm in  $L^2(\mathbb{T}^d)$ :  $|u|_2^2 = \int_{\mathbb{T}^d} dx |u|^2(x, t)$ . We have that

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 = -\frac{1}{\beta^2} |\nabla u|_2^2 - \frac{1}{\beta} \int_{\mathbb{T}^d} dx u(x, t) \nabla u(x, t) \nabla (U \star h)(x, t). \tag{5.2}$$

Since  $h \in L^1(\mathbb{T}^d)$  and  $\nabla U$  is bounded

$$\sup_{x,t} |\nabla (U \star h)(x, t)| \leq \bar{c}.$$

Then, for any  $\delta > 0$

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 \leq -\frac{1}{\beta^2} |\nabla u|_2^2 + \frac{\bar{c}}{\beta} |\nabla u|_2 |u|_2 \leq -(1-\delta) \frac{1}{\beta^2} |\nabla u|_2^2 + \frac{1}{4\delta} \bar{c} |u|_2^2. \quad (5.3)$$

By Gronwall there exists a constant  $C$  such that

$$|u|_2^2 \leq |\bar{u}|_2^2 e^{Ct},$$

so that

$$\int_0^T dt |u(t)|_2^2 \leq C |\bar{u}|_2^2, \quad \int_0^T |\nabla u|_2^2 \leq C |\bar{u}|_2^2$$

for some constant  $C$ . Here and below  $C$  denotes a running constant. Moreover,

$$\|u'\|_{-1} = \sup_{v \in H^1: \|v\|_1=1} \left\{ -\int_{\mathbb{T}^d} \nabla v \left[ \frac{1}{\beta} \nabla U \star h \right] + \frac{1}{\beta^2} \nabla u \right\} \leq \frac{\bar{c}}{\beta} |u|_2 + \frac{1}{\beta^2} |\nabla u|_2.$$

Hence

$$\int_0^T dt \|u'(t)\|_{-1}^2 \leq C |\bar{u}|_2^2.$$

Consider now functions  $u : \mathbb{T}^d \rightarrow \mathbb{R}^2$ . We define the Hilbert space  $W$  in this case as before, simply using as scalar product  $(\cdot, \cdot)$ , the scalar product in  $L^2(\mathbb{T}^d; \mathbb{R}^2)$ . We use the same notation for  $W$  and  $W_1$ . We say that  $\rho = (\rho_1, \rho_2)$  is a weak solution of (1.6) if for all  $v \in H^1(\mathbb{T}^d; \mathbb{R}^2)$  and a.e.  $0 \leq t \leq T$

$$\beta^2 (v, \rho'_i) + (\nabla v, \nabla \rho_i + \beta \rho_i \nabla U \star \rho_j) = 0$$

and  $\rho(\cdot, 0) = \rho(\cdot)$ .

Theorem 5.1 defines a map  $A$  from  $L^2(0, T; L^2(\mathbb{T}^d; \mathbb{R}^2))$  in itself by applying it to a set of two equations for  $u_i, i = 1, 2$  with a given term depending on  $g_i, i = 1, 2$  in the following way:

$$\begin{aligned} \beta^2 \partial_t u_i &= \Delta u_i + \beta \nabla (u_i \nabla (U \star g_j)), \\ u_i(\cdot, 0) &= \bar{u}_i(\cdot), \\ i, j &= 1, 2, \quad i \neq j. \end{aligned} \quad (5.4)$$

We use  $g = (g_1, g_2)$  and  $u = (u_1, u_2)$ ,  $|g|_2^2 = \sum_{i=1,2} |g_i|_2^2$ . Then, since the  $L^1$  norm of  $g_j$  is bounded by the  $L^2$  norm, namely  $L^1([0, T], \mathbb{T}^d) \in L^2([0, T], \mathbb{T}^d)$ , there exists a solution  $u$  in  $W$ , and we can write

$$\begin{aligned} A(g) &= u, \\ \|A(g)\|_W^2 &\leq C |\bar{u}|_2^2. \end{aligned}$$

We now prove the existence theorem for the nonlinear set of equations by proving that  $A$  is continuous and maps a closed convex set in a compact set.

*Compactness.* We consider the closed and convex set  $X \in L^2([0, T], \mathbb{T}^d)$ :

$$X = \{h : \|h\|_{L^2(0,T;L^2)}^2 \leq k\}.$$

Since  $A(h)$  is in  $W$  and  $W$  is compactly imbedded in  $L^2([0, T], \mathbb{T}^d)$  the image of  $X$  is compact.

*Continuity.* Consider  $g, \tilde{g} \in L^2(0, T; L^2)$ . Let  $u = A(g)$  and  $\tilde{u} = A(\tilde{g})$  be the corresponding weak solutions. We have that, for  $i \neq j$ ,

$$\begin{aligned} (u_i - \tilde{u}_i, u'_i - \tilde{u}'_i) &= -\frac{1}{\beta^2} \int_{\mathbb{T}^d} |\nabla (u_i - \tilde{u}_i)|^2 - \frac{1}{\beta} \int_{\mathbb{T}^d} (u_i - \tilde{u}_i) \nabla (u_i - \tilde{u}_i) \cdot \nabla (U \star g_j) \\ &\quad - \frac{1}{\beta} \int_{\mathbb{T}^d} \tilde{u}_i \nabla (u_i - \tilde{u}_i) \cdot \nabla U \star (g_j - \tilde{g}_j), \end{aligned} \quad (5.5)$$

$$\frac{1}{2} \frac{d}{dt} |u_i - \tilde{u}_i|_2^2 \leq -C |\nabla (u_i - \tilde{u}_i)|_2^2 + c_1 |u_i - \tilde{u}_i|_2^2 + c_2 |g_j - \tilde{g}_j|_2^2.$$

We obtained the previous inequality using that the  $L^1$  norm of  $(g - \tilde{g})$  is bounded by the  $L^2$  norm. Therefore,

$$\|u - \tilde{u}\|_{L^2([0, T], L^2)} \leq C \|g - \tilde{g}\|_{L^2([0, T], L^2)}$$

which proves the continuity of  $A$  in  $L^2([0, T], L^2)$ .

By Schauder's theorem the map  $A$  has a fixed point in  $L^2([0, T], L^2)$  which is the weak solution we were looking for.

*Uniqueness.* The proof is standard [GL].  $\square$

Summarizing, we have proved the following theorem.

**Theorem 5.2.** *There exists a unique weak solution in  $W_1$  to the following Cauchy problem*

$$\begin{aligned} \beta^2 \partial_t \rho_i &= \Delta \rho_i + \beta \nabla (\rho_i \nabla (U \star \rho_j)), \\ \rho_i(\cdot, 0) &= \bar{\rho}_i(\cdot), \\ i, j &= 1, 2, \quad i \neq j. \end{aligned} \tag{5.6}$$

*Regularity.* If  $\nabla U \star \rho \in C^0([0, T]; C^1)$  and  $\bar{\rho} \in C^2(\mathbb{T}^d)$  then the linear equation has a classical solution. Since the weak solution  $\rho$  is also in  $C^0([0, T]; L^2)$  we have that, indeed,  $\nabla U \star \rho \in C^0([0, T]; C^1)$  and therefore the weak solution  $\rho$  corresponding to an initial datum in  $C^2(\mathbb{T}^d)$  is a classical solution.

## 6. Sharp interface limit

In this section we study the solutions of (1.1) in the sharp interface limit in a three-dimensional torus  $\Omega$ . We introduce again the scale separation parameter  $\varepsilon$ , which has the meaning of ratio between the kinetic and macroscopic scales. Then, we scale position and time as  $\varepsilon^{-1}$  and  $\varepsilon^{-3}$ , respectively, while keeping fixed (equal to 1)  $\gamma$ . The width of the interface on the macroscopic scale is then of order  $\varepsilon$ , so that in the limit  $\varepsilon \rightarrow 0$  the interface becomes sharp. The rescaled density distributions  $f_i^\varepsilon(r, v, t) = f_i(\varepsilon^{-1}r, v, \varepsilon^{-3}t)$  are solutions of

$$\begin{aligned} \partial_t f_i^\varepsilon + \varepsilon^{-2} v \cdot \nabla_r f_i^\varepsilon + \varepsilon^{-2} F_i^\varepsilon \cdot \nabla_v f_i^\varepsilon &= \varepsilon^{-3} L_\beta f_i^\varepsilon, \\ F_i^\varepsilon(r, t) &= -\nabla_r \int dr' \varepsilon^{-3} U(\varepsilon^{-1}|r - r'|) \int dv' f_j^\varepsilon(r', v', t) =: -\nabla_r g_i^\varepsilon. \end{aligned} \tag{6.1}$$

In this section  $F_i^\varepsilon$  depends on  $\varepsilon$  through the function  $f_j^\varepsilon$  but also through the potential since we are keeping fixed  $\gamma$ . We consider a situation in which initially an interface is present. Since the stationary nonhomogeneous solutions of (1.1) are given by the Maxwellian multiplied by the front density profiles we let our system start initially close to those stationary solutions and choose as initial datum  $f_i^\varepsilon(r, v) = M_\beta(v) \rho_i^\varepsilon$ , where the density profiles are very close to a profile such that in the bulk its values are  $\rho_i^\pm$ , the values of the densities in the two pure phases at temperature  $T$ , and the interpolation between them on the interface is realized along the normal direction in each point by the fronts. We put  $\rho_i^\pm = \bar{\rho}^\pm$  and use the symmetry properties of the segregation phase transition giving  $\rho_2^\mp = \bar{\rho}^\pm$ . Consider a smooth surface  $\Gamma_0 \subset \Omega$ . Let  $d(r, \Gamma_0)$  be the signed distance of the point  $r \in \Omega$  from the interface. Consider an initial profile for the densities  $\rho_i^\varepsilon$  of the following type: at distance greater than  $O(\varepsilon)$  from the interface (in the bulk) the density profiles  $\rho_i^\varepsilon(r)$  are almost constant equal to  $\rho_i^\pm$ ; at distance  $O(\varepsilon)$  (near the interface) we choose

$$\rho_i^\varepsilon(r) = w_i(\varepsilon^{-1}d(r, \Gamma_0)) + O(\varepsilon), \tag{6.2}$$

where  $w_i(z)$  are the fronts, which are one-dimensional solutions of (1.4) with asymptotic values  $\rho_i^\pm$ . Since these solutions are unique up to a translation we fix a solution by imposing that  $w_1(0) = w_2(0)$ .

Let  $\Gamma_t^\varepsilon$  be an interface at time  $t$  defined by

$$\Gamma_t^\varepsilon = \{r \in \Omega : \rho_1^\varepsilon(r, t) = \rho_2^\varepsilon(r, t)\}$$

and  $T$  be such that  $\Gamma_t^\varepsilon$  is regular for  $t \in [0, T]$ . Let  $d^\varepsilon(r, t)$  be the signed distance  $d(r, \Gamma_t^\varepsilon)$  of  $r \in \Omega$  from the interface  $\Gamma_t^\varepsilon$ , such that  $d^\varepsilon > 0$  in  $\Omega_t^{\varepsilon,+}$  and  $d^\varepsilon < 0$  in  $\Omega_t^{\varepsilon,-}$ , where  $\Omega = \Gamma_t^\varepsilon \cup \Omega_t^{\varepsilon,+} \cup \Omega_t^{\varepsilon,-}$ . For the sake of simplicity we drop from now on the apex  $\varepsilon$ . For any  $r$  such that  $|d(r, t)| < 1/k(\Gamma_t)$ ,  $k(\Gamma_t) = \sup_{x \in \Gamma_t} k(x)$  with  $k(x)$  the maximum of the principal curvatures in  $x$ , there exists  $s(r) \in \Gamma_t$  such that

$$v(s(r))d(r, t) + s(r) = r,$$

where  $v(s(r))$  is the normal to the surface  $\Gamma_t$  in  $s(r)$ . Hence,

$$v(s(r)) = \nabla_r d(r, t), \quad r \in \Gamma_t.$$

Define the normal velocity of the interface as

$$V(s(r)) = \partial_t d(r, t).$$

The curvature  $K$  (the sum of the principal curvatures) is given by  $K = \Delta_r d(r, t)$ ,  $r \in \Gamma_t$ . Define, for  $\varepsilon^0$  small enough,

$$\mathcal{N}(\delta) := \{r : |d(r, t)| < \delta\}$$

where  $\delta = 1/n$ ,  $n = \max_{t \in [0, T], 0 \leq \varepsilon \leq \varepsilon^0} k(\Gamma_t)$ .

We follow the approach based on the truncated Hilbert expansions introduced by Caffish [C]. This method, which has been used in the previous section to prove the hydrodynamic limit for the Vlasov–Fokker–Planck equation, has been improved by including boundary layer expansions in [ELM], to prove the hydrodynamic limit for the Boltzmann equation in a slab. Here we try to adapt the arguments in [ELM] to the fact that the boundary is not given *a priori* and has to be found as a result of the expansion. The Hilbert expansion is nothing but a power expansion in  $\varepsilon$  for the solution of the kinetic equation

$$f^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n f^{(n)}. \quad (6.3)$$

Since we expect that the behaviour of the solution will be different in the bulk and near the interface, we decompose  $f^{(n)}$  into two parts: the bulk part  $\hat{f}^{(n)}(r, t)$  and boundary terms  $\tilde{f}^{(n)}$  which will be fast varying functions close to the interface, namely, they depend on  $r, t$  in the following way:

$$\tilde{f}^{(n)} = \tilde{f}^{(n)}(\varepsilon^{-1}d(r, t), r, t),$$

while  $\hat{f}^{(n)}(r, t)$  are slowly varying functions on the microscopic scale. More precisely, a fast varying function  $h(r, t)$  for  $r \in \mathcal{N}$  can be represented as a function  $h(z, r, t)$ ,  $z = \varepsilon^{-1}d(r, t)$ , with the condition  $h(z, r + \ell v(s(r)), t) = h(z, r, t)$ ,  $\forall \ell$  small enough. Hence in  $\mathcal{N}$  we can write

$$\nabla_r h = \frac{1}{\varepsilon} v \partial_z h + \bar{\nabla}_r h, \quad \partial_t h = \frac{1}{\varepsilon} V \partial_z h + \bar{\partial}_t h, \quad \Delta_r h = \frac{1}{\varepsilon^2} \partial_z^2 h + \frac{1}{\varepsilon} (\nabla_r \cdot v) \partial_z h + \bar{\Delta}_r h, \quad (6.4)$$

where the bar on the derivative operators means derivatives with respect to  $r$ , keeping the other variables fixed. Note that  $v \cdot \bar{\nabla}_r h(z, r, t) = 0$ .

To write the expansion for the force term  $F_i^\varepsilon$  we introduce  $U^\varepsilon \star \sum_{n=0}^{\infty} \varepsilon^n \rho_j^{(n)} = \sum_{n=0}^{\infty} \varepsilon^n g_i^{(n)}$  and  $F_i^{(n)} = -\nabla_r g_i^{(n)}$ . We expand also the signed distance

$$d(r, t) = \sum_{n=0}^{\infty} \varepsilon^n d^{(n)}(r, t). \quad (6.5)$$

We will denote by  $v^{(n)}$  the gradient  $\nabla_r d^{(n)}$ , with  $\bar{v} := v^{(0)}$ . The condition  $|\nabla_r d|^2 = 1$  is equivalent to

$$\begin{aligned} |\nabla_r d^{(0)}|^2 &= 1, & \nabla_r d^{(0)} \nabla_r d^{(1)} &= 0, \\ \nabla_r d^{(0)} \nabla_r d^{(j)} &= -\frac{1}{2} \sum_{i=1}^{j-1} \nabla_r d^{(i)} \nabla_r d^{(j-i)}, & j &\geq 2, \end{aligned}$$

so that  $d^{(0)}$  can be interpreted as a signed distance from an interface that we denote by  $\bar{\Gamma}_t$ . As a consequence of (6.5) the velocity of the interface  $\Gamma_t$  has the form

$$V = \sum_{i=0}^{\infty} \varepsilon^i V^{(i)}, \quad \bar{V} := V^{(0)}.$$

We remark that velocity  $V$  determines the curve evolving with it. Velocity  $\bar{V}$  will generate an order zero interface  $\bar{\Gamma}_t$ . The interface generated by  $\sum_i \varepsilon^i V^{(i)}$  will be a deformation, small for small  $\varepsilon$ , of  $\bar{\Gamma}_t$ . We define

$$\begin{aligned} \mathcal{N}^0(m) &:= \{r : |d^{(0)}(r, t)| < m\}, & \bar{\Gamma}_t &:= \{r : |d^{(0)}(r, t)| = 0\}, \\ \Omega^{\pm(-)} &:= \{r : |d^{(0)}(r, t)| > (<)0\} \end{aligned}$$

and fix  $m$  so that  $\mathcal{N}^0(m) \subset \mathcal{N}(\delta)$ .

We assume that in  $\Omega^\pm \setminus \mathcal{N}^0(m)$

$$f^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \hat{f}^{(n)} \tag{6.6}$$

and that in  $\mathcal{N}^0(m)$ , the solution is of the form

$$f^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \tilde{f}^{(n)}. \tag{6.7}$$

We will match the inner and outer expansions in  $z = \varepsilon^{-1}m$  with  $m = \varepsilon^c$ ,  $c \in (0, 1)$ . Hence, we require that as  $z \rightarrow \pm\infty$  [CF]

$$\begin{aligned} \tilde{f}_i^{(0)} &= (\hat{f}_i^{(0)})^\pm + \mathcal{O}(e^{-\alpha|z|}), \\ \tilde{f}_i^{(1)} &= (\hat{f}_i^{(1)})^\pm + v^{(0)} \cdot (\nabla_r \hat{f}_i^{(0)})^\pm (z - d^{(1)}) + \mathcal{O}(e^{-\alpha|z|}), \\ \tilde{f}_i^{(2)} &= (\hat{f}_i^{(2)})^\pm + v^{(0)} \cdot (\nabla_r \hat{f}_i^{(1)})^\pm (z - d^{(1)}) + (\nabla_r \hat{f}_i^{(0)})^\pm \cdot (-v^{(0)} d^{(2)} + v^{(1)}(z - d^{(1)})) \\ &\quad + \frac{1}{2} (\partial_{r_h} \partial_{r_k} \hat{f}_i^{(0)})^\pm v_h^{(0)} (z - d^{(1)}) v_k^{(0)} (z - d^{(1)}) + \mathcal{O}(e^{-\alpha|z|}), \\ \tilde{f}_i^{(3)} &= (\hat{f}_i^{(3)})^\pm + v^{(0)} \cdot (\nabla_r \hat{f}_i^{(2)})^\pm (z - d^{(1)}) + (\nabla_r \hat{f}_i^{(1)})^\pm \cdot (-v^{(0)} d^{(2)} + v^{(1)}(z - d^{(1)})) \\ &\quad + \frac{1}{2} (\partial_{r_h} \partial_{r_k} \hat{f}_i^{(1)})^\pm v_h^{(0)} v_k^{(0)} (z - d^{(1)})^2 + (\nabla_r \hat{f}_i^{(0)})^\pm \cdot (v^{(2)}(z - d^{(1)}) - v^{(0)} d^{(3)} - v^{(1)} d^{(2)}) \\ &\quad + (\partial_{r_h} \partial_{r_k} \hat{f}_i^{(0)})^\pm v_h^{(0)} (z - d^{(1)}) (-v_k^{(0)} d^{(2)} + v_k^{(1)}(z - d^{(1)})) \\ &\quad + \frac{1}{6} (\partial_{r_h} \partial_{r_k} \partial_{r_l} \hat{f}_i^{(0)})^\pm (v_h^{(0)} v_k^{(0)} v_l^{(0)} (z - d^{(1)})^3) + \mathcal{O}(e^{-\alpha|z|}), \\ &\quad \dots \end{aligned}$$

where the symbol  $(h)^\pm$  for the hat functions stands for  $\lim_{\ell \rightarrow 0^\pm} h(r + v\ell)$ ,  $r \in \bar{\Gamma}_t$  and the same for the derivatives. We replace (6.6) and (6.7) in the equations and equate terms of the same order in  $\varepsilon$  separately in  $\Omega^\pm \setminus \mathcal{N}^0(m)$  and  $\mathcal{N}^0(m)$ . We will use the notation  $\rho_i^{(n)} = \int dv f_i^{(n)}$ , and we denote by  $\hat{h}, \tilde{h}$  a function  $h(f_i^{(n)})$  whenever it is evaluated on  $\hat{f}_i^{(n)}, \tilde{f}_i^{(n)}$ .

*Outer expansion*In  $\Omega^\pm \setminus \mathcal{N}^0(m)$ ,  $n \geq 0$ 

$$\partial_t \hat{f}_i^{(n-3)} + v \cdot \nabla_r \hat{f}_i^{(n-1)} + \sum_{l,l' \geq 0: l+l'=n-1} \hat{F}_i^{(l)} \cdot \nabla_v \hat{f}_i^{(l')} = L_\beta \hat{f}_i^{(n)}, \quad (6.8)$$

with  $\hat{f}_i^\alpha = 0$ ,  $\alpha < 0$ .*Inner expansion*In  $\mathcal{N}^0(m)$   $n \geq 0$  we have

$$\begin{aligned} \sum_{l,l' \geq 0: l+l'=n-2} V^{(l')} \partial_z \tilde{f}_i^{(l)} + \sum_{k+k'=n} v^{(k)} \cdot v \partial_z \tilde{f}_i^{(k')} + v \cdot \bar{\nabla}_r \tilde{f}_i^{(n-1)} + \bar{\partial}_t \tilde{f}_i^{(n-3)} \\ - \sum_{l,l',l'' \geq 0: l+l'+l''=n} \partial_z \tilde{g}_i^{(l)} v^{(l')} \cdot \nabla_v \tilde{f}_i^{(l'')} - \sum_{l,l' \geq 0: l+l'=n-1} \bar{\nabla}_r \tilde{g}_i^{(l)} \cdot \nabla_v \tilde{f}_i^{(l')} = L_\beta \tilde{f}_i^{(n)}, \end{aligned} \quad (6.9)$$

with  $\tilde{f}_i^\alpha = 0$ ,  $\alpha < 0$ .

The strategy for a rigorous proof is to construct, once the functions  $f_i^{(n)}$  have been determined, the solution in terms of a truncated Hilbert expansion as

$$f^\varepsilon = \sum_{n=0}^N \varepsilon^n f^{(n)} + \varepsilon^m R, \quad (6.10)$$

where the functions are evaluated in  $z = \varepsilon^{-1} d^N(r, t)$ , with  $d^N(r, t) = \sum_{i=0}^{N-2} \varepsilon^i d^{(i)}(r, t)$  and then write a weakly nonlinear equation for the remainder. In this approach it is essential to have enough smoothness for the terms of the expansion. In contrast, they would be discontinuous on the border of  $\mathcal{N}^0(m)$  since  $\tilde{f}^{(n)}$  are not exactly equal to  $\hat{f}^{(n)}$  there, but differ for terms exponentially small in  $\varepsilon$ . One can modify the expansion terms by interpolating, in a smooth way, between the outside and the inside, getting smooth terms, which do not satisfy the equations for terms exponentially small in  $\varepsilon$ , that can be put in the remainder. With this in mind, we did not put in the equations the terms coming from the force such that in the convolution  $r$  is in  $\mathcal{N}^0(m)$  and  $r'$  in  $\Omega^\pm \setminus \mathcal{N}^0(m)$ . That is possible because the potential is of finite range. Finally, we remark that the terms  $f_i^{(n)}$  of the expansion do not depend on  $\varepsilon$  but they will be evaluated in  $z = \varepsilon^{-1} d^N(r, t)$ , which depends on  $\varepsilon$  because of the rescaling and also because the interface at time  $t$  still depends on  $\varepsilon$ . The latter is a new feature in the framework of the Hilbert expansion due to the fact that the boundary is not fixed but is itself unknown.

In this section we show how to construct the terms  $f_i^{(n)}$ . The argument is formal because we do not prove boundedness of the remainder nor the regularity properties of the terms of the expansion. We plan to report on that in the future.

Now we go back to the Hilbert power series and start examining the equations order by order. We will find explicitly only the first three terms in the expansion to explain the procedure.

*Outer expansion*At the lowest order  $\varepsilon^{-3}$  ( $n = 0$ )

$$L_\beta \hat{f}_i^{(0)} = 0,$$

which implies that  $\hat{f}_i^{(0)}$  has to be Maxwellian in velocity with variance  $T$  times a function  $\hat{\rho}_i^{(0)}(r, t)$ . The latter is found by looking at the equations at the next two orders. At order  $\varepsilon^{-2}$  ( $n = 1$ )

$$v \cdot \nabla_r \hat{f}_i^{(0)} + \hat{F}_i^{(0)} \cdot \nabla_v \hat{f}_i^{(0)} = L_\beta \hat{f}_i^{(1)}. \quad (6.11)$$

The solution is of the form

$$\hat{f}_i^{(1)} = \hat{\rho}_i^{(1)} M_\beta - M_\beta \hat{\rho}_i^{(0)} v \cdot \nabla_r \hat{\mu}_i^{(0)}, \quad (6.12)$$

where  $\mu_i^\varepsilon(\rho^\varepsilon) = T \ln \rho_i^\varepsilon + U^\varepsilon \star \rho_j^\varepsilon$  and  $\mu_i^\varepsilon = \sum_{n=0}^\infty \varepsilon^n \mu_i^{(n)}$ .

The order  $\varepsilon^{-1}$  equation ( $n = 2$ ) is

$$v \cdot \nabla_r \hat{f}_i^{(1)} + \hat{F}_i^{(0)} \cdot \nabla_v \hat{f}_i^{(1)} + \hat{F}_i^{(1)} \cdot \nabla_v \hat{f}_i^{(0)} = L_\beta \hat{f}_i^{(2)}. \quad (6.13)$$

The solvability condition for this equation says that the integral on the velocity of the lhs has to be zero. By integrating over the velocity and using the explicit expression for  $\hat{f}_i^{(1)}$  we get

$$-T \nabla_r \cdot (\hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(0)}) = 0.$$

Hence the solvability condition for the equation  $n = 2$  gives the equation determining  $\hat{\rho}_i^{(0)}$ . The choice of the initial data implies that the only solution of that equation is the constant one, with values  $\rho_i^\pm$  in  $\Omega^\pm$ . We look at the next order  $n = 3$  to find  $\hat{\rho}_i^{(1)}$  by the solvability condition. By integrating over  $v$  the equation  $n = 3$  and taking into account that  $\hat{f}_i^{(0)}$  is Maxwellian in velocity, we get the following condition on  $\hat{u}^{(2)}$ , where  $u_i^{(n)} = \int dv v f_i^{(n)}$ ,

$$\nabla_r \cdot \hat{u}_i^{(2)} = 0. \quad (6.14)$$

Then,  $\hat{f}_i^{(2)}$  is determined, by replacing (6.12) in equation (6.13), as

$$\hat{f}_i^{(2)} = -M_\beta \hat{\rho}_i^{(0)} v \cdot \nabla_r \hat{\mu}_i^{(1)} + \hat{\rho}_i^{(2)} M_\beta, \quad (6.15)$$

where  $\hat{\mu}_i^{(1)} = T \hat{\rho}_i^{(1)} / \hat{\rho}_i^{(0)} + \hat{g}_i^{(1)}$ .

We use  $\hat{f}_i^{(2)}$  as given by (6.15) to get  $\hat{u}_i^{(2)} = -T \hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(1)}$  and plug it in (6.14) to get the equation for  $\hat{\mu}_i^{(1)}$ :

$$\Delta_r \hat{\mu}_i^{(1)} = 0.$$

We consider equation (6.8) for  $n = 3$ :

$$\partial_t \hat{f}_i^{(0)} + v \cdot \nabla_r \hat{f}_i^{(2)} + \sum_{l, l' \geq 0: l+l'=2} \hat{F}_i^{(l)} \cdot \nabla_v \hat{f}_i^{(l')} = L_\beta \hat{f}_i^{(3)}, \quad (6.16)$$

whose solution is

$$\hat{f}_i^{(3)} = -M_\beta v \cdot [\hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(2)} + \hat{\rho}_i^{(1)} \nabla_r \hat{\mu}_i^{(1)}] + M_\beta \frac{T}{2} \hat{\rho}_i^{(0)} (v \cdot \nabla_r)(v \cdot \nabla_r) \hat{\mu}_i^{(1)} + \hat{\rho}_i^{(3)} M_\beta. \quad (6.17)$$

The equation for  $\hat{\mu}_i^{(2)} = T \hat{\rho}_i^{(2)} / \hat{\rho}_i^{(0)} - T/2 (\hat{\rho}_i^{(1)} / \hat{\rho}_i^{(0)})^2 + \hat{g}_i^{(2)}$  comes from the equation for  $n = 4$ :

$$\partial_t \hat{f}_i^{(1)} + v \cdot \nabla_r \hat{f}_i^{(3)} - \sum_{l, l' \geq 0: l+l'=3} \nabla_r \hat{g}_i^{(l)} \cdot \nabla_v \hat{f}_i^{(l')} = L_\beta \hat{f}_i^{(4)},$$

which gives as solvability condition  $\nabla_r \cdot \hat{u}_i^{(3)} = -\partial_t \hat{\rho}_i^{(1)}$  where  $\hat{u}_i^{(3)} = \int dv \hat{f}_i^{(3)}$ . Using (6.17) we get

$$\Delta \hat{\mu}_i^{(2)} = \frac{1}{T \hat{\rho}_i^{(0)}} \partial_t \hat{\rho}_i^{(1)} - \frac{\nabla_r \hat{\rho}_i^{(1)} \cdot \nabla_r \hat{\mu}_i^{(1)}}{\hat{\rho}_i^{(0)}} := \frac{S_i}{\hat{\rho}_i^{(0)}}, \quad S_i = \beta \partial_t \hat{\rho}_i^{(1)} - \nabla_r \hat{\rho}_i^{(1)} \cdot \nabla_r \hat{\mu}_i^{(1)}.$$

*Inner expansion*

At the lowest order ( $n = 0$ )

$$v \cdot \bar{v} \partial_z \tilde{f}_i^{(0)} - \bar{v} \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(0)}.$$

In appendix A it is proved that any solution of this equation has the form  $M_\beta(v) \tilde{\rho}_i^{(0)}$ , with  $\tilde{\rho}_i^{(0)}$  a function of  $z$ . Plugging back in the equation we have

$$\partial_z \tilde{\rho}_i^{(0)} + \beta \tilde{\rho}_i^{(0)} \partial_z (\tilde{U} \star \tilde{\rho}_j^{(0)}) = 0 \iff \partial_z \tilde{\mu}_i^{(0)} = 0, \quad (6.18)$$

where  $\tilde{U}$  is the potential  $U$  integrated over all coordinates but one. We solve this equation with the conditions at infinity  $\rho_i^\pm$ , given by the matching conditions, and call  $w_i$  the front solution.

The exponential decay of  $w_i$  has been proved for the one-component case [DOPT] and the same argument should provide the proof also in this case. We can conclude that in  $\Omega$

$$f_i^{(0)}(r, t) = M_\beta \left[ w \left( \frac{d(r, t)}{\varepsilon} \right) \chi_m + (1 - \chi_m) \hat{\rho}_i^{(0)} \right],$$

with  $\chi_m$  the characteristic function of  $\mathcal{N}^0(m)$ . This solution differs from the front solution  $w_i$  in  $\Omega$  for terms which are exponentially small in  $\varepsilon$  and has the disadvantage of not being continuous on the border of  $\mathcal{N}^0$ . As explained before, it has to be modified as

$$f_i^{(0)}(r, t) = M_\beta \left[ w \left( \frac{d(r, t)}{\varepsilon} \right) h(d(r, t)) + (1 - h(d(r, t))) \hat{\rho}_i^{(0)}(r, t) \right]$$

with  $h$  a smooth version of  $\chi_m$ .

We now find  $\tilde{f}_i^{(1)}$  by examining the  $\varepsilon^{-2}$  order ( $n = 1$ ):

$$v \cdot \bar{v} \partial_z \tilde{f}_i^{(1)} - \bar{v} \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(1)} - \bar{v} \cdot \nabla_v \tilde{f}_i^{(1)} \partial_z \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(1)}. \quad (6.19)$$

The term involving  $v^{(1)}$ ,  $v^{(1)} \cdot (v \partial_z \tilde{f}_i^{(0)} - \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(0)}) = \beta v \cdot v^{(1)} M_\beta \hat{\rho}_i^{(0)} \partial_z \tilde{\mu}_i^{(0)} = 0$ , because  $\tilde{f}_i^{(0)}$  is solution of the lowest order equation and the bar operators vanish because  $\hat{\rho}_i^{(0)}$  is a function of  $z$  only. In appendix A we show that the solution is necessarily Maxwellian in velocity so that we can write  $\tilde{f}_i^{(1)} = \tilde{\rho}_i^{(1)} M_\beta$  with  $\tilde{\rho}_i^{(1)}$  to be determined by the following equation:

$$\partial_z \tilde{\rho}_i^{(1)} + \beta \tilde{\rho}_i^{(0)} \partial_z \tilde{g}_i^{(1)} + \beta \tilde{\rho}_i^{(1)} \tilde{U} \star \partial_z \tilde{\rho}_j^{(0)} = 0. \quad (6.20)$$

Taking into account that  $-\beta \tilde{U} \star \partial_z \tilde{\rho}_j^{(0)} = \partial_z \ln w_i$ , from the equation for the front, we get

$$\partial_z (T \tilde{\rho}_i^{(1)} (w_i)^{-1} + \tilde{g}_i^{(1)}) = 0 \iff \partial_z \tilde{\mu}_i^{(1)} = 0. \quad (6.21)$$

Hence, the value of  $\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)}$  in  $z = 0$  is enough to find  $\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)}$  for any  $z$ . This value is found as follows. From

$$\tilde{\mu}_i^{(1)} = T (\tilde{\rho}_i^{(1)}) (w_i)^{-1} + \tilde{U} \star \tilde{\rho}_j^{(1)} + \bar{K} \int dz' (z - z') \tilde{U}(z - z') w_j(z'),$$

where  $\bar{K} = \Delta_r d^{(0)}(r, t)$  is the curvature of the interface  $\bar{\Gamma}_t$  (see appendix C), we want to find  $\tilde{\rho}_i^{(1)}$  as determined by  $\tilde{\mu}_i^{(1)}$ . We define the operator  $\mathcal{L}$  as  $(\mathcal{L}h)_i = T h_i (w_i)^{-1} + \tilde{U} \star h_j$ . The previous relation reads as

$$(\mathcal{L}\tilde{\rho}^{(1)})_i = \tilde{\mu}_i^{(1)} - \bar{K} \int dz' (z - z') \tilde{U}(z - z') w_j(z'). \quad (6.22)$$

The operator  $\mathcal{L}$  has a zero mode since  $\mathcal{L}w' = 0$ , so that the equation  $(\mathcal{L}\tilde{\rho}^{(1)})_i = h_i$  has a solution only if

$$\sum_{i=1,2} \int dz h_i(z) w'_i(z) = 0.$$

The solvability condition for (6.22) is

$$\sum_{i=1,2} \int dz \tilde{\mu}_i^{(1)} w'_i(z) = \bar{K} \sum_{i=1,2} \int dz dz' w'_i(z) (z - z') \tilde{U}(z - z') w_j(z'), \quad (6.23)$$

which implies because  $\tilde{\mu}_i^{(1)}$  are constant:

$$\tilde{\mu}_1^{(1)}(0, r, t) [w_1]_{-\infty}^{+\infty} + \tilde{\mu}_2^{(1)}(0, r, t) [w_2]_{-\infty}^{+\infty} = \bar{K}(r, t) \sum_{i,i \neq j} \int dz dz' w'_i(z) (z - z') \tilde{U}(z - z') w_j(z').$$

In appendix B it is shown that the sum on the right-hand side (rhs) is the surface tension  $S$  for this model, so we have (since  $[w_1]_{-\infty}^{+\infty} = -[w_2]_{-\infty}^{+\infty}$ )

$$(\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)})(0, r, t)[w_1]_{-\infty}^{+\infty} = \bar{K}(r, t)S. \quad (6.24)$$

The matching conditions impose that  $\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)} \rightarrow (\hat{\mu}_1^{(1)})^\pm - (\hat{\mu}_2^{(1)})^\pm$  for  $z \rightarrow \pm\infty$ , so that for  $r \in \bar{\Gamma}_t$

$$[(\hat{\mu}_1^{(1)})^\pm - (\hat{\mu}_2^{(1)})^\pm][w_1]_{-\infty}^{+\infty} = \bar{K}(r, t)S, \quad (6.25)$$

and hence the continuity of  $\hat{\mu}_1^{(1)} - \hat{\mu}_2^{(1)}$  on the interface.

The conservation law for the equation at the order  $\varepsilon^{-1}$  ( $n = 2$ ) will give the velocity of the interface. By integrating this equation over velocity we get

$$w'_i \bar{V} + \partial_z(\bar{v} \cdot \tilde{u}_i^{(2)}) = 0, \quad (6.26)$$

where the fact that  $\tilde{f}_i^{(0)}$  and  $\tilde{f}_i^{(1)}$  are Maxwellian in velocity is crucial for several cancellations. By integrating over  $z$

$$-\bar{V}[w_i]_{-\infty}^{+\infty} = [\bar{v} \cdot \tilde{u}_i^{(2)}]_{-\infty}^{+\infty}. \quad (6.27)$$

By the matching conditions  $\tilde{u}_i^{(2)} \rightarrow (\hat{u}_i^{(2)})^\pm$  at  $\pm\infty$ , so that for  $r \in \bar{\Gamma}_t$

$$-\bar{V}[w_i]_{-\infty}^{+\infty} = [\bar{v} \cdot \hat{u}_i^{(2)}]_{\pm}^{\pm}, \quad r \in \bar{\Gamma}_t. \quad (6.28)$$

Summarizing what we got so far we have constructed functions  $\hat{\mu}_i^{(1)}$  harmonic in  $\Omega^\pm$  which satisfy (6.25) and (6.28). For the sake of convenience we will denote by  $\bar{\mu}_i^{(1)}$  the functions defined in  $\Omega$ , not necessarily smooth, equal to  $(\hat{\mu}_i^{(1)})^\pm$  in  $\Omega \setminus \bar{\Gamma}_t$  and such that  $\lim_{d^{(0)}(r,t) \rightarrow 0^\pm} \bar{\mu}_i^{(1)} = (\hat{\mu}_i^{(1)})^\pm|_{\bar{\Gamma}_t}$  and the same for the derivatives. This means that  $\bar{\mu}_i^{(1)}$  satisfy

$$\begin{aligned} \Delta \bar{\mu}_i^{(1)} &= 0, & r &\in \Omega \setminus \bar{\Gamma}_t, \\ (\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})[\bar{\rho}^+ - \bar{\rho}^-] &= \bar{K}(r, t)S, & r &\in \Omega \setminus \bar{\Gamma}_t, \\ \bar{V}[w_i]_{-\infty}^{+\infty} &= [T \bar{\rho}_i \bar{v} \cdot \nabla_r \bar{\mu}_i^{(1)}]_{\pm}^{\pm}, & r &\in \bar{\Gamma}_t, \end{aligned} \quad (6.29)$$

where  $\bar{\rho}^\pm = w_1(\pm\infty)$ . Let us write the last equation as

$$\bar{V}\beta[\bar{\rho}^+ - \bar{\rho}^-] = [(\bar{\rho} + \bar{\varphi})\bar{v} \cdot \nabla_r \bar{\mu}_1^{(1)}]_{\pm}^{\pm} = [(\bar{\varphi} - \bar{\rho})\bar{v} \cdot \nabla_r \bar{\mu}_2^{(1)}]_{\pm}^{\pm} \quad (6.30)$$

and

$$\bar{\rho}(r) = \frac{\bar{\rho}_1(r) + \bar{\rho}_2(r)}{2}, \quad \bar{\varphi}(r) = \frac{\bar{\rho}_1(r) - \bar{\rho}_2(r)}{2},$$

with  $\bar{\rho}_i(r)$  the step functions  $\bar{\rho}_i(r) := \bar{\rho}_i^+ \chi^+ + \bar{\rho}_i^- \chi^-$ ,  $\chi^\pm$  the characteristic functions of the sets  $d^{(0)}(r, t) > 0$  and  $d^{(0)}(r, t) < 0$ , respectively. We know, because of the symmetry of the phase transition, that  $\bar{\rho}$  is constant while  $\bar{\varphi}$  is discontinuous in 0 and  $\bar{\varphi}(r) = \pm|\bar{\varphi}|$  for  $r \in \Omega^\pm$ . The previous equation implies

$$2\bar{V}\beta[\bar{\rho}^+ - \bar{\rho}^-] = \bar{\rho}[\bar{v} \cdot \nabla_r(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})]_{\pm}^{\pm} + [\bar{\varphi}\bar{v} \cdot \nabla_r(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})]_{\pm}^{\pm}, \quad (6.31)$$

$$0 = \bar{\rho}[\bar{v} \cdot \nabla_r(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})]_{\pm}^{\pm} + [\bar{\varphi}\bar{v} \cdot \nabla_r(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})]_{\pm}^{\pm}. \quad (6.32)$$

We introduce the function  $\zeta(r, t) = (\bar{\rho}_1 \bar{\mu}_1^{(1)} + \bar{\rho}_2 \bar{\mu}_2^{(1)})(r, t) = \bar{\rho}(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)}) + \bar{\varphi}(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})$  so that  $\Delta_r \zeta(r, t) = 0$  in  $\Omega \setminus \bar{\Gamma}_t$  and (6.32) gives  $[\bar{v} \cdot \nabla_r \zeta]_{\pm}^{\pm} = 0$ . Moreover, it is discontinuous on  $\bar{\Gamma}_t$  because of the function  $\bar{\varphi}$ . The jump is

$$\zeta^+(r, t) - \zeta^-(r, t) = 2|\bar{\varphi}|(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}), \quad r \in \bar{\Gamma}_t.$$

In conclusion,  $\zeta$  satisfies

$$\begin{aligned}\Delta_r \zeta(r, t) &= 0 & r \in \Omega \setminus \bar{\Gamma}_t, \\ [\zeta]_{-}^{+} &= 2|\bar{\varphi}|S\bar{K}(r, t)/[w_1]_{-\infty}^{+\infty} & r \in \bar{\Gamma}_t, \\ 0 &= [\bar{v} \cdot \nabla_r \zeta]_{-}^{+} & r \in \bar{\Gamma}_t.\end{aligned}\quad (6.33)$$

It is possible to show using the Green identity that this problem for a given function  $\bar{K}(r, t)$  has the unique solution

$$\begin{aligned}\zeta(r, t) &= \int_{\bar{\Gamma}_t} ds (\zeta^{+} - \zeta^{-})(s, t) v \cdot \nabla G(r, s) = \frac{2S|\bar{\varphi}|}{[w_1]_{-\infty}^{+\infty}} \int_{\bar{\Gamma}_t} ds \bar{K}(s, t) v \cdot \nabla G(r, s), \\ & r \in \Omega \setminus \bar{\Gamma}_t \\ \frac{(\zeta^{+} + \zeta^{-})}{2}(r, t) &= \frac{2S|\bar{\varphi}|}{[w_1]_{-\infty}^{+\infty}} \int_{\bar{\Gamma}_t} ds \bar{K}(s, t) \bar{v} \cdot \nabla G(r, s), & r \in \bar{\Gamma}_t,\end{aligned}$$

where  $G$  is the Green function in  $\Omega$ . We note that  $(\zeta^{+} + \zeta^{-}) = 2\bar{\rho}(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})$ .

We consider now the function  $\xi(r, t) = (\bar{\rho}_1 \bar{\mu}_1^{(1)} - \bar{\rho}_2 \bar{\mu}_2^{(1)})(r, t) = \bar{\rho}(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}) + \bar{\varphi}(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})$ , which is discontinuous on  $\bar{\Gamma}_t$  and satisfies

$$\begin{aligned}\Delta_r \xi(r, t) &= 0 & r \in \Omega \setminus \bar{\Gamma}_t, \\ [\xi]_{-}^{+} &= \frac{|\bar{\varphi}|}{\bar{\rho}} (\zeta^{+} + \zeta^{-}) & r \in \bar{\Gamma}_t, \\ \bar{V} &= \frac{T}{2} \frac{[\bar{v} \cdot \nabla_r \xi]_{-}^{+}}{[\bar{\rho}^{+} - \bar{\rho}^{-}]} & r \in \bar{\Gamma}_t.\end{aligned}\quad (6.34)$$

The problem is well posed because given the current configuration of the front the problem has a unique solution and this solution in turn determines the velocity of the front.

In conclusion we have determined  $\bar{\mu}_1^{(1)}$  and  $\bar{\mu}_2^{(1)}$ . In  $\mathcal{N}^0(m)$   $\tilde{\mu}_i^{(1)}$  is a constant equal to the value  $\bar{\mu}_i^{(1)}(r, t)$ ,  $r \in \bar{\Gamma}_t$ , which is determined by solving the limiting equation. Hence  $\tilde{\mu}_i^{(1)}$  and  $\hat{\mu}_i^{(1)}$  are known at this stage. As a consequence,  $\hat{\rho}_i^{(1)}$  are known through the relation  $\hat{\mu}_i^{(1)} = T(\hat{\rho}_i^{(1)}/\bar{\rho}_i) + \hat{U}\hat{\rho}_i^{(1)}$  in  $\Omega \setminus \mathcal{N}^0(m)$  while  $\tilde{\rho}_i^{(1)}$  are found as solutions of (6.22) with the rhs decaying to a constant as  $z \rightarrow \pm\infty$ , and the decay is exponential if  $w_i$  does so. Then, a modification of the argument in [CCO1] leads to the exponential decay of  $\tilde{\rho}_i^{(1)}$ . We note that  $\tilde{\rho}_i^{(1)}$  is determined by (6.22) but for a term  $\alpha w_i'$  which is in the null of  $\mathcal{L}$ , with  $\alpha$  independent of  $z$ . To fix  $\alpha$  it is enough to put the condition  $\tilde{\rho}_1^{(1)}(0, r, t) = \tilde{\rho}_2^{(1)}(0, r, t)$ ,  $r \in \mathcal{N}^0(m)$ . Since we have fixed  $\rho_1^\varepsilon = \rho_2^\varepsilon$  on  $\Gamma^\varepsilon$  we are allowed to choose  $\tilde{\rho}_1^{(k)}(0, r, t) = \tilde{\rho}_2^{(k)}(0, r, t)$ ,  $r \in \mathcal{N}^0(m)$  for any  $k$ .

We proceed now constructing the higher orders of the expansion. For  $n = 2$

$$\begin{aligned}\bar{V} \partial_z \tilde{f}_i^{(0)} + \bar{v} \cdot v \partial_z \tilde{f}_i^{(2)} + v \cdot \bar{\nabla}_r \tilde{f}_i^{(1)} - \partial_z \tilde{g}_i^{(2)} \bar{v} \cdot \nabla_v \tilde{f}_i^{(0)} - \partial_z \tilde{g}_i^{(0)} \bar{v} \cdot \nabla_v \tilde{f}_i^{(2)} \\ - \partial_z \tilde{g}_i^{(1)} \bar{v} \cdot \nabla_v \tilde{f}_i^{(1)} + \bar{\nabla}_r \tilde{g}_i^{(1)} \cdot \nabla_v \tilde{f}_i^{(0)} = L_\beta \tilde{f}_i^{(2)}.\end{aligned}\quad (6.35)$$

Again, the terms involving  $v^{(2)}$  and  $v^{(1)}$  are zero thanks to the previous equations. The matching conditions require for  $z$  large

$$\tilde{f}_i^{(2)}(\pm|z|, r, t) = (\hat{f}_i^{(2)})^\pm + \bar{v} \cdot (\nabla_r \hat{f}_i^{(1)})^\pm (z - d^{(1)}) + O(e^{-\alpha|z|}).$$

Hence, we have to solve a stationary problem on the real line with given conditions at infinity. We replace in (6.35)  $\tilde{f}_i^{(2)} = \tilde{q}_i^{(2)} + \tilde{\rho}_i^{(2)} M_\beta$  with  $\int dv \tilde{q}_i^{(2)} = 0$ . This means that  $\tilde{q}_i^{(2)}$  is in the orthogonal to the kernel of  $L_\beta$  versus the scalar product

$$(f, g)_{M_\beta} = \int dv M_\beta^{-1} f g.$$

We have

$$M_\beta[\bar{V}\partial_z w_i + \bar{v} \cdot v \partial_z \tilde{\rho}_i^{(2)} + v \cdot \bar{\nabla}_r \tilde{\rho}_i^{(1)} + \beta \partial_z \tilde{g}_i^{(2)} \bar{v} \cdot v w_i + \beta \partial_z \tilde{g}_i^{(0)} \bar{v} \cdot v \tilde{\rho}_i^{(2)} + \beta \partial_z \tilde{g}_i^{(1)} \bar{v} \cdot v \tilde{\rho}_i^{(1)} - \beta v \cdot \bar{\nabla}_r \tilde{g}_i^{(1)} w_i] = L_\beta \tilde{q}_i^{(2)} - \bar{v} \cdot v \partial_z \tilde{q}_i^{(2)} + \partial_z \tilde{g}_i^{(0)} \bar{v} \cdot \nabla_v \tilde{q}_i^{(2)}. \quad (6.36)$$

Using the equation for the front  $w_i$  and the fact that  $\partial_z \tilde{\mu}_i^{(1)} = 0$  together with  $\tilde{\mu}_i^{(2)} = (T/w_i) \tilde{\rho}_i^{(2)} - T/2(\tilde{\rho}_i^{(1)}/w_i)^2 + \tilde{g}_i^{(2)}$  we get

$$\begin{aligned} M_\beta \bar{v} \cdot v [\partial_z \tilde{\rho}_i^{(2)} + \beta w_i \partial_z \tilde{g}_i^{(2)} + \beta \tilde{\rho}_i^{(2)} \partial_z \tilde{g}_i^{(0)} + \beta \partial_z \tilde{g}_i^{(1)} \tilde{\rho}_i^{(1)}] \\ = M_\beta \bar{v} \cdot v w_i \partial_z \left[ \frac{\tilde{\rho}_i^{(2)}}{w_i} + \beta \tilde{g}_i^{(2)} - \frac{\beta}{2} \left( \frac{\tilde{\rho}_i^{(1)}}{w_i} \right)^2 \right] = \beta M_\beta \bar{v} \cdot v w_i \partial_z \tilde{\mu}_i^{(2)}. \end{aligned}$$

Hence we can write equation (6.36) in the form

$$\begin{aligned} \beta M_\beta \bar{v} \cdot v w_i \partial_z \tilde{\mu}_i^{(2)} = L_\beta \tilde{q}_i^{(2)} - \bar{v} \cdot v \partial_z \tilde{q}_i^{(2)} + \partial_z \tilde{g}_i^{(0)} \bar{v} \cdot \nabla_v \tilde{q}_i^{(2)} + \beta M_\beta w_i v \cdot \bar{\nabla}_r \tilde{g}_i^{(1)} \\ - M_\beta v \cdot \bar{\nabla}_r \tilde{\rho}_i^{(1)} - M_\beta \bar{V} \partial_z w_i. \end{aligned} \quad (6.37)$$

From (6.15) the conditions at infinity are

$$\tilde{f}_i^{(2)}(\pm|z|, r, t) = M_\beta[(\hat{\rho}_i^{(2)})^\pm - (\hat{\rho}_i^{(0)})^\pm v \cdot (\nabla_r \hat{\mu}_i^{(1)})^\pm + \bar{v} \cdot (\nabla_r \hat{\rho}_i^{(1)})^\pm (z - d^{(1)})] + O(e^{-\alpha|z|}).$$

As a consequence

$$\begin{aligned} \tilde{\rho}_i^{(2)}(\pm|z|, r, t) &= (\hat{\rho}_i^{(2)})^\pm(r, t) + \bar{v} \cdot (\nabla_r \hat{\rho}_i^{(1)})^\pm(z - d^{(1)}) + O(e^{-\alpha|z|}), \\ \beta \bar{v} \cdot \tilde{u}_i^{(2)}(\pm|z|, r, t) &= -(\hat{\rho}_i^{(0)})^\pm \bar{v} \cdot (\nabla_r \hat{\mu}_i^{(1)})^\pm + O(e^{-\alpha|z|}), \\ \lim_{z \rightarrow \pm\infty} \int dv P(v) \tilde{q}_i^{(2)}(z, r, t) &= -\langle v P(v) \rangle_\beta \cdot (\nabla_r \hat{\mu}_i^{(1)})^\pm (\hat{\rho}_i^{(0)})^\pm, \end{aligned}$$

where  $P(v)$  is a polynomial in the velocity and  $\langle \cdot \rangle_\beta$  are the moments of the Maxwellian  $M_\beta$ :  $\langle v P(v) \rangle_\beta = \int dv v P(v) M_\beta$ . The matching conditions for the chemical potential imply that as  $z \rightarrow \pm\infty$

$$\tilde{\mu}_i^{(2)} = (\hat{\mu}_i^{(2)})^\pm + (z - d^{(1)}) \bar{v} \cdot (\nabla_r \hat{\mu}_i^{(1)})^\pm + O(e^{-\alpha|z|}) := (\hat{\mu}_i^{(2)})^\pm + C_i^\pm.$$

We have that

$$w_i(+\infty)(\hat{\mu}_i^{(2)})^+ - w_i(-\infty)(\hat{\mu}_i^{(2)})^- = [w_i(\tilde{\mu}_i^{(2)} - C_i)]_{-\infty}^{+\infty},$$

where  $C_i = 1_{z < 0} C_i^- + 1_{z > 0} C_i^+$ . The lhs can be written as

$$\int dz \partial_z [w_i(\tilde{\mu}_i^{(2)} - C_i)] = \int dz [w_i \partial_z (\tilde{\mu}_i^{(2)} - C_i) + w_i' (\tilde{\mu}_i^{(2)} - C_i)].$$

We multiply (6.37) by  $v_z = \bar{v} \cdot v$  and integrate over  $v$

$$w_i \partial_z \tilde{\mu}_i^{(2)} = -\partial_z \int dv v_z^2 \tilde{q}_i^{(2)} - \beta \bar{v} \cdot \tilde{u}_i^{(2)}, \quad (6.38)$$

where we used  $\tilde{q}_i^{(2)} \in [\text{Ker } L_\beta]^\perp$ ,  $\bar{v} \cdot \bar{\nabla}(\cdot) = 0$  and  $L_\beta \tilde{q}_i^{(2)} = L_\beta \tilde{f}_i^{(2)}$ . We have also

$$w_i \partial_z (\tilde{\mu}_i^{(2)} - C_i) = -\partial_z \int dv v_z^2 \tilde{q}_i^{(2)} - \beta \bar{v} \cdot \tilde{u}_i^{(2)} - w_i \partial_z C_i. \quad (6.39)$$

Remembering the matching conditions for  $\tilde{u}_i^{(2)}$ , the fact that  $\int dv v_z^2 \tilde{q}_i^{(2)}$  vanishes at infinity and since

$$\partial_z C_i = 1_{z < 0} (\bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)})^- + 1_{z > 0} (\bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)})^+$$

we have that asymptotically in  $|z|$

$$\beta \bar{v} \cdot \tilde{u}_i^{(2)} + w_i \partial_z C_i = O(e^{-\alpha|z|}),$$

so that the integral over  $z$  of the rhs of (6.39) makes sense. Then

$$\begin{aligned} w_i(+\infty)(\hat{\mu}_i^{(2)})^+ - w_i(-\infty)(\hat{\mu}_i^{(2)})^- \\ = - \int dz [\beta \bar{v} \cdot \tilde{u}_i^{(2)} + w_i \partial_z C_i] + \int dz w'_i \tilde{\mu}_i^{(2)} - \int dz w'_i C_i. \end{aligned} \quad (6.40)$$

Moreover the function  $\bar{v} \cdot \tilde{u}_i^{(2)}$  is known because

$$\bar{V} w'_i + \partial_z \bar{v} \cdot \tilde{u}_i^{(2)} = 0.$$

It follows that

$$\bar{v} \cdot \tilde{u}_i^{(2)} = \frac{w_i - \bar{\rho}^-}{\bar{\rho}^+ - \bar{\rho}^-} \left[ \left( \frac{\hat{\rho}_i^{(0)}}{\beta} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)} \right)^- - \left( \frac{\hat{\rho}_i^{(0)}}{\beta} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)} \right)^+ \right] - \left( \frac{\hat{\rho}_i^{(0)}}{\beta} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)} \right)^-.$$

So only the last term involving  $\tilde{\mu}_i^{(2)}$  in (6.40) needs to be computed. But we know that

$$\tilde{\mu}_i^{(2)} = \frac{T}{w_i} \tilde{\rho}_i^{(2)} - \frac{T}{2} \left( \frac{\tilde{\rho}_i^{(1)}}{w_i} \right)^2 + \tilde{g}_i^{(2)} = \mathcal{L} \tilde{\rho}_i^{(2)} + B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)}), \quad (6.41)$$

where  $B_1$  and  $B_2$  are defined in appendix C. The solvability condition is

$$\sum_i \int_{-\infty}^{+\infty} dz [\tilde{\mu}_i^{(2)} - D_i] w'_i = 0, \quad (6.42)$$

where  $D_i := B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)})$ . Summing on  $i = 1, 2$

$$\sum_i \int dz w'_i \tilde{\mu}_i^{(2)} = \sum_i \int_{-\infty}^{+\infty} dz D_i w'_i = \sum_i \int_{-\infty}^{+\infty} dz w'_i (B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)})). \quad (6.43)$$

We introduce functions  $\bar{\mu}_i^{(2)}$  defined as explained after (6.28). We have

$$[\bar{\rho}_1 \bar{\mu}_1^{(2)} + \bar{\rho}_2 \bar{\mu}_2^{(2)}]_{\pm}^+ = \sum_i \left[ - \int dz [\beta \bar{v} \cdot \tilde{u}_i^{(2)} + w_i \partial_z C_i] + \int dz w'_i (D_i - C_i) \right] =: H. \quad (6.44)$$

We are now in a position to find the first correction to the velocity of the interface,  $V^{(1)}$ . This is given by the solvability condition for the boundary equation for  $n = 3$ :

$$\begin{aligned} \sum_{l,l' \geq 0; l+l'=1} V^{(l)} \partial_z \tilde{f}_i^{(l)} + \sum_{l,l' \geq 0; l+l'=3} v \cdot v^{(l)} \partial_z \tilde{f}_i^{(l')} + v \cdot \bar{\nabla}_r \tilde{f}_i^{(2)} + \bar{\partial}_t \tilde{f}_i^{(0)} \\ - \sum_{l,l',l'' \geq 0; l+l'+l''=3} \partial_z \tilde{g}_i^{(l)} v^{(l'')} \cdot \nabla_v \tilde{f}_i^{(l')} - \sum_{l,l' \geq 0; l+l'=2} \bar{\nabla}_r \tilde{g}_i^{(l)} \cdot \nabla_v \tilde{f}_i^{(l')} = L_{\beta} \tilde{f}_i^{(3)}. \end{aligned}$$

After integration on  $v$  we get

$$\bar{\nabla}_r \cdot \tilde{u}_i^{(2)} + \bar{V} \partial_z \tilde{\rho}_i^{(1)} + V^{(1)} \partial_z w_i + \partial_z (\bar{v} \cdot \tilde{u}_i^{(3)}) + \partial_z (v^{(1)} \cdot \tilde{u}_i^{(2)}) = 0.$$

Taking into account that for  $z$  large

$$\begin{aligned} \bar{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z \bar{v} \cdot \tilde{u}_i^{(3)} &= (\bar{\nabla}_r \cdot \hat{u}_i^{(2)})^{\pm} + \bar{v} \cdot (\nabla_r (\bar{v} \cdot \hat{u}_i^{(2)}))^{\pm} + O(e^{-\alpha|z|}) \\ &= (\nabla_r \cdot \hat{u}_i^{(2)})^{\pm} + O(e^{-\alpha|z|}) = O(e^{-\alpha|z|}), \end{aligned}$$

because  $\nabla_r \cdot \hat{u}_i^{(2)} = 0$ , we can integrate over  $z$ :

$$\bar{V} [\tilde{\rho}_i^{(1)}]_{-\infty}^{+\infty} + V^{(1)} [w_i]_{-\infty}^{+\infty} = - \int dz [\bar{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z (\bar{v} \cdot \tilde{u}_i^{(3)})] - [v^{(1)} \cdot \tilde{u}_i^{(2)}]_{-\infty}^{+\infty}. \quad (6.45)$$

By the matching conditions we have as  $z \rightarrow \pm\infty$

$$\tilde{u}_i^{(3)} \sim (\hat{u}_i^{(3)})^{\pm} + (z - d^{(1)}) \bar{v} \cdot (\nabla_r \hat{u}_i^{(2)})^{\pm}.$$

Let us introduce the functions

$$n_i = 1_{z < 0}(z - d^{(1)})\bar{v} \cdot (\nabla_r(\bar{v} \cdot \hat{u}_i^{(2)}))^- + 1_{z > 0}(z - d^{(1)})\bar{v} \cdot (\nabla_r(\bar{v} \cdot \hat{u}_i^{(2)}))^+$$

so that

$$\partial_z n_i = 1_{z < 0}\bar{v} \cdot (\nabla_r(\bar{v} \cdot \hat{u}_i^{(2)}))^- + 1_{z > 0}\bar{v} \cdot (\nabla_r(\bar{v} \cdot \hat{u}_i^{(2)}))^+.$$

Thus

$$\begin{aligned} \int dz[\bar{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z(\bar{v} \cdot \tilde{u}_i^{(3)})] &= \int dz[\bar{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z n_i + \partial_z(\bar{v} \cdot \tilde{u}_i^{(3)} - n_i)] \\ &= \int dz[\bar{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z n_i] + [\bar{v} \cdot \hat{u}_i^{(3)}]_-^+ = A_i + [\bar{v} \cdot \hat{u}_i^{(3)}]_-^+, \end{aligned}$$

where we still have to compute  $A_i$ . Essentially we need to know  $\bar{\nabla}_r \cdot \tilde{u}_i^{(2)}$ . It can be derived from (6.37) multiplying for  $\bar{v}_\perp \cdot v$ , where  $\bar{v}_\perp$  denotes one of the two directions orthogonal to  $\bar{v}$ , and integrating in  $v$ . After some computations we have

$$\beta \bar{v}_\perp \cdot \tilde{u}_i^{(2)} = -\partial_z \int dv(\bar{v}_\perp \cdot v)(\bar{v} \cdot v)\tilde{q}_i^{(2)} + w_i \bar{v}_\perp \cdot \bar{\nabla}_r \tilde{g}_i^{(1)} + \frac{1}{\beta} \bar{v}_\perp \cdot \bar{\nabla}_r \tilde{\rho}_i^{(1)}.$$

In order to obtain the overlined divergence of  $\tilde{u}_i^{(2)}$  we have to sum over the two orthogonal directions; then remembering the asymptotic behaviour of  $\int dv P(v)\tilde{q}_i^{(2)}$ , we can conclude that

$$A_i = \int dz \left[ \frac{1}{\beta} w_i \bar{\nabla}_r \cdot \bar{\nabla}_r \tilde{g}_i^{(1)} + \frac{1}{\beta^2} \bar{\nabla}_r \cdot \bar{\nabla}_r \tilde{\rho}_i^{(1)} + \partial_z n_i \right].$$

Here we recall that  $\hat{u}_i^{(2)}$  depends only on quantities of the previous order in  $\varepsilon$ , which are known. Moreover by (6.17) we have also

$$\hat{u}_i^{(3)} = -T[\bar{\rho}_i \bar{v} \cdot \nabla_r \hat{\mu}_i^{(2)} + \bar{\rho}_i^{(1)} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)}]$$

so that

$$\begin{aligned} \frac{1}{\beta} [\hat{\rho}_i^{(0)} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(2)}]_-^+ &= \bar{V}[\hat{\rho}_i^{(1)}]_-^+ + V^{(1)}[\hat{\rho}_i^{(0)}]_-^+ - \frac{1}{\beta} [\hat{\rho}_i^{(1)} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)}]_-^+ + A_i \\ &\quad - \frac{1}{\beta} [\hat{\rho}_i^{(0)} v^{(1)} \cdot \nabla_r \hat{\mu}_i^{(1)}]_-^+. \end{aligned} \quad (6.46)$$

We consider the functions  $\zeta^{(2)} = \bar{\rho}_1 \bar{\mu}_1^{(2)} + \bar{\rho}_2 \bar{\mu}_2^{(2)}$  and  $\xi^{(2)} = \bar{\rho}_1 \bar{\mu}_1^{(2)} - \bar{\rho}_2 \bar{\mu}_2^{(2)}$ . We have from (6.44)

$$[\zeta^{(2)}]_-^+ = H(r, t),$$

and by summing in equation (6.46)

$$\begin{aligned} [\bar{v} \cdot \nabla_r \zeta^{(2)}]_-^+ &= \beta \bar{V}[\hat{\rho}_1^{(1)} + \hat{\rho}_2^{(1)}]_-^+ + \sum_i \{-[\hat{\rho}_i^{(1)} \bar{v} \cdot \nabla_r \hat{\mu}_i^{(1)}]_-^+ + \beta A_i - [\hat{\rho}_i^{(0)} v^{(1)} \cdot \nabla_r \hat{\mu}_i^{(1)}]_-^+\} \\ &:= P. \end{aligned}$$

Moreover, we have the identity  $[\xi^{(2)}]_-^+ = \zeta^{(2)+} + \zeta^{(2)-}$ .

We get the velocity  $V^{(1)}$  by taking the difference in (6.46) on the index  $i$

$$\begin{aligned} 2[\bar{\rho}^+ - \bar{\rho}^-]V^{(1)} &= \frac{1}{\beta} [\bar{v} \cdot \nabla_r \xi^{(2)}]_-^+ - \bar{V}[\hat{\rho}_1^{(1)} - \hat{\rho}_2^{(1)}]_-^+ + \frac{1}{\beta} [\hat{\rho}_1^{(1)} \bar{v} \cdot \nabla_r \hat{\mu}_1^{(1)}]_-^+ \\ &\quad - \frac{1}{\beta} [\hat{\rho}_2^{(1)} \bar{v} \cdot \nabla_r \hat{\mu}_2^{(1)}]_-^+ - A_1 + A_2 + \frac{1}{\beta} [\hat{\rho}_1^{(0)} v^{(1)} \cdot \nabla_r \hat{\mu}_1^{(1)}]_-^+ - \\ &\quad - \frac{1}{\beta} [\hat{\rho}_2^{(0)} v^{(1)} \cdot \nabla_r \hat{\mu}_2^{(1)}]_-^+ := 2[\bar{\rho}^+ - \bar{\rho}^-]Q. \end{aligned}$$

Note that  $\int_{\Gamma_t} V^{(1)}$  is not necessarily zero as was  $\int_{\Gamma_t} \bar{V}$ . This implies that the volume enclosed by the interface  $\Gamma_t^{(1)}$  evolving with  $V^{(0)} + \varepsilon V^{(1)}$  is not conserved.

In conclusion,  $\xi^{(2)}$  and  $\zeta^{(2)}$  are solutions of

$$\begin{aligned} \Delta_r \xi^{(2)}(r, t) &= (S_1 - S_2)(r, t) & r \in \Omega \setminus \bar{\Gamma}_t, \\ [\xi^{(2)}]_{\pm}^+ &= \frac{|\bar{\varphi}|}{\bar{\rho}} (\zeta^{(2)+} + \zeta^{(2)-}) & r \in \bar{\Gamma}_t, \\ V^{(1)} &= Q & r \in \bar{\Gamma}_t \end{aligned} \quad (6.47)$$

and

$$\begin{aligned} \Delta_r \zeta^{(2)}(r, t) &= (S_1 + S_2)(r, t) & r \in \Omega \setminus \bar{\Gamma}_t, \\ [\zeta^{(2)}]_{\pm}^+ &= H(r, t) & r \in \bar{\Gamma}_t, \\ [\bar{v} \cdot \nabla_r \zeta^{(2)}]_{\pm}^+ &= P(r, t) & r \in \bar{\Gamma}_t, \end{aligned} \quad (6.48)$$

$S_i$ ,  $Q$  have been determined before. The terms  $H$  and  $P$  depend on  $d^{(0)}$  which has been already found and also on  $d^{(1)}$  which is unknown and has to be determined by  $V^{(1)} = \partial_t d^{(1)}$ . These equations are different from the first order equations because the surface  $\bar{\Gamma}_t$  is given, so that we are not facing a free boundary problem. In this sense they are ‘linearized’ even if the equations remain nonlinear. The problem is well-posed because given  $d^{(1)}$  on  $\bar{\Gamma}_t$  the problem has a unique solution and this solution in turn determines velocity  $V^{(1)}$ . Then,  $d^{(1)}$  is found in  $\mathcal{N}^0(m)$  through the condition  $\nabla d^{(1)} \nabla d^{(0)} = 0$ . The argument is analogous to the one in [ABC].

Once  $\bar{\mu}_i^{(2)}$  are found as solutions of these equations we can find  $\hat{\rho}_i^{(2)}$  in terms of  $\hat{\mu}_i^{(2)}$ . We have now the asymptotic values needed to solve (6.37). This equation admits a solution if the conditions at infinity satisfy suitable conditions. The matching conditions require that the solution at infinity grows linearly. This is a problem analogous to the so-called Kramers problem in the half space [BCN]. It can be reduced to a Riemann problem with fixed asymptotic values at infinity in the following way:

Since the solution  $\tilde{f}_i^{(2)}$  has to be approximately  $A_i^{\pm} + z B_i^{\pm}$  at infinity, the functions  $A_i^{\pm} = M_{\beta}((\hat{\rho}_i^{(2)})^{\pm} - v \cdot (\hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(1)})^{\pm})$  and  $B_i^{\pm} = M_{\beta}(v \cdot \nabla_r \hat{\rho}_i^{(1)})^{\pm}$  have to satisfy

$$v \cdot v(B_i^{\pm} + M_{\beta} \beta \hat{U}(\hat{\rho}_i^{(0)} \bar{v} \cdot \nabla_r \hat{\rho}_i^{(1)})^{\pm}) + M_{\beta} v \cdot (\hat{\rho}_i^{(0)} \bar{\nabla}_r \hat{\mu}_i^{(1)})^{\pm} = L_{\beta}(A_i^{\pm} + z B_i^{\pm}),$$

because all the other terms in (6.35) vanish at infinity. This is equivalent to

$$L_{\beta}(B_i^{\pm}) = 0, \quad L_{\beta} A_i^{\pm} = \bar{v} \cdot v(B_i^{\pm} + M_{\beta} \beta \hat{U}(\hat{\rho}_i^{(0)} v \cdot \nabla_r \hat{\rho}_i^{(1)})^{\pm}) + M_{\beta} v \cdot (\hat{\rho}_i^{(0)} \bar{\nabla}_r \hat{\mu}_i^{(1)})^{\pm}.$$

This is true by direct inspection. Then, the problem is reduced to a well-posed problem of finding a solution  $h_i$  to equation (6.35) decaying to  $M_{\beta}(\hat{\rho}_i^{(2)})^{\pm}$  at infinity.

Similar arguments lead to the computation of higher order terms.

We conclude this section by remarking that our expansion is different from the one in [CCO1] which is more similar to a Chapman–Enskog expansion because the terms of their expansion  $f_i^{(n)}$  still depend on  $\varepsilon$  and are determined by equations which are nonlinear in the interface at every order in the sense that they are free-boundary problems determining for any  $n$  an interface  $\Gamma^{(n)}$  moving with velocity  $\sum_{i=0}^n \varepsilon^i V^{(i)}$ . Our approach is still based on the matching conditions and is in a way intermediate between [ABC] and [Yu], where the hydrodynamic limit for the Boltzmann equation in the presence of shocks is proved by constructing a Hilbert expansion to approximate the solution.

## 7. Interface motion

In this section we discuss the equations for the interface motion. We start by rewriting them in terms of the quantities  $\zeta$  and  $\psi = \bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}$ . The equation for  $\psi$  is similar to the Mullins–Sekerka equation but for the fact that there is an extra term determining the velocity

$$\begin{aligned} \Delta_r \psi(r, t) &= 0 & r \in \Omega \setminus \Gamma_t, \\ \psi(r, t) &= \frac{S\bar{K}(r, t)}{\bar{\rho}^+ - \bar{\rho}^-} & r \in \Gamma_t, \\ V &= \frac{T}{2[\bar{\rho}^+ - \bar{\rho}^-]} \left[ \frac{1}{\bar{\rho}} (\bar{\rho}^2 - |\bar{\varphi}|^2) [\bar{v} \cdot \nabla_r \psi]_{\pm}^+ + \frac{1}{\bar{\rho}} [\bar{\varphi} \bar{v} \cdot \nabla_r \zeta]_{\pm}^+ \right] & r \in \Gamma_t. \end{aligned} \quad (7.1)$$

The jump of  $\bar{\varphi} \bar{v} \cdot \nabla_r \zeta$  in the last term on the rhs is indeed  $2|\bar{\varphi}|v \cdot \nabla_r \zeta(r, t)$ ,  $r \in \Gamma_t$  and

$$\begin{aligned} \Delta_r \zeta(r, t) &= 0 & r \in \Omega \setminus \Gamma_t, \\ [\zeta]_{\pm}^+ &= 2|\bar{\varphi}|S\bar{K}(r, t)/[w_1]_{\pm\infty}^{+\infty} & r \in \Gamma_t, \\ 0 &= [\bar{v} \cdot \nabla_r \zeta]_{\pm}^+ & r \in \Gamma_t. \end{aligned} \quad (7.2)$$

Hence there are two contributions to the velocity of the interface:  $V_{\text{MS}}$ , the velocity of an interface in the Mullins–Sekerka motion, and  $V_{\text{HS}}$ , the velocity of an interface in the two-phase Hele–Shaw motion (7.2). The latter describes the motion of a bubble of gas expanding into a fluid in a radial Hele–Shaw cell and is a free-boundary problem for the pressure  $P$

$$\begin{aligned} \Delta_r P(r, t) &= 0 & r \in \Omega \setminus \Gamma_t, \\ [p]_{\pm}^+ &= CK(r, t)/[w_1]_{\pm\infty}^{+\infty} & r \in \Gamma_t, \\ V &= v \cdot \nabla_r P & r \in \Gamma_t. \end{aligned} \quad (7.3)$$

Equations (7.1) and (7.2) are identical to the equations in [OE], describing the sharp interface motion for the dynamics of incompressible fluid mixtures driven by thermodynamic forces, modelling a polymer blend. In this paper the macroscopic equation is a modification of the Cahn–Hilliard equation for a mixture of two fluids that includes a Lagrangian multiplier  $p$  (‘pressure’) to take into account the constraint of constant total density:

$$\begin{aligned} \partial_t \rho_i &= \nabla \cdot (\rho_i \nabla (\mu_i + p)), & i = 1, 2, \\ \rho_1 + \rho_2 &= 1. \end{aligned} \quad (7.4)$$

This produces in the macroscopic equation for the concentration a convective term which in turn gives rise to the Hele–Shaw contribution  $V_{\text{HS}}$  to the interface motion. The macroscopic equations (1.6) with  $\mu_i = \delta\mathcal{F}/\delta\rho_i$  differ from the ones above for the constraint and hence for the pressure term. It is easy to see that the formal sharp interface limit is the same for both equations with a divergence-free field  $\nabla\zeta$  appearing as a velocity field in the equation for the total density which is constant in the bulk at the first order. Moreover, thermodynamic relations give that  $\nabla\zeta = \nabla p^{(1)}$  with  $p^{(1)}$  the first correction to the effective pressure. Hence, the role of  $\nabla\zeta$  is exactly the same as the Lagrangian multiplier  $p$  in [OE].

We remark that the Hele–Shaw motion has more conserved quantities than the Mullins–Sekerka motion. In fact, the former conserves the volume of each connected component of both phases, while the latter conserves only the total volume as we can easily see by starting from

$$\frac{d}{dt} |\Omega_{\Gamma}^+| = \int_{\Gamma} V,$$

where  $\Omega_{\Gamma}^+$  is the region enclosed in the surface  $\Gamma$ . We consider now a situation in which there are  $N$  closed curves  $\Gamma_i$  dividing  $\Omega$  into  $N$  connected components  $\Omega_{\Gamma_i}^+$ . In the Mullins–Sekerka

problem the velocity is proportional to the jump of the normal derivative of the harmonic function  $f$  and this implies using the divergence theorem

$$\sum_i \frac{d}{dt} |\Omega_{\Gamma_i}^+| = \sum_i \int_{\Gamma_i} [v \cdot \nabla f]_-^+ = \sum_i \int_{\Omega_{\Gamma_i}^+} \Delta f + \int_{\Omega^-} \Delta f = 0,$$

where  $\Omega^-$  is the complement of  $\bigcup_i (\Omega_{\Gamma_i}^+ \cup \Gamma_i)$ . In the Hele–Shaw problem the velocity is proportional to the normal derivative of  $f$  and

$$\frac{d}{dt} |\Omega_{\Gamma_i}^+| = \int_{\Gamma_i} v \cdot \nabla f = \int_{\Omega_{\Gamma_i}^+} \Delta f = 0.$$

In the problem (7.1), (7.2) this fact has consequences on the evolution of the droplets of the two phases. The relative importance of the two contributions  $V_{\text{HS}}$  and  $V_{\text{MS}}$  is ruled by the coefficients: if  $(\bar{\rho}^-)^{-1} - (\bar{\rho}^+)^{-1} \ll 1$  (near the critical point) the  $V_{\text{MS}}$  term dominates, while for deep quenches the  $V_{\text{HS}}$  term prevails. We refer to [OE] for the discussion on the behaviour of the interface as given in (7.1) and (7.2) and its relevance for the behaviour of a polymer blend.

Here we want to just comment on the applicability of our model to polymer blend. We are modelling in the mean field limit a mixture of two species of interacting particles whose velocities undergo a Brownian motion. We have in mind situations in which the transport of energy and momentum are not relevant, so that the dynamics does not conserve momentum or energy. In the diffusive limit we obtain, as shown in sections 2–4, a continuous description of a mixture of two species of interacting particles whose position is subject to Brownian motion. The nonlocal free energy of our model in this limit is given by (1.5) (with  $\gamma = 1$ ) that we write in a form more suited for this discussion as

$$\begin{aligned} \mathcal{F}(\rho_1, \rho_2) = & \int dx [(\rho_1 \ln \rho_1)(x) + (\rho_2 \ln \rho_2)(x) + \beta \alpha \rho_1(x) \rho_2(y)] \\ & - \frac{1}{2} \int dx dy U(x-y) [\rho_1(x) - \rho_1(y)] [\rho_2(x) - \rho_2(y)] \end{aligned} \quad (7.5)$$

with  $\alpha = \int U(x-y)$ . We can think instead of chains of polymers of two types,  $A$  and  $B$ , each chain subject to Brownian motion modelling the interaction of the chain with the environment. In the mean field approach the nonlocal free energy for a mixture of polymers is given by the Flory–Huggins–de Gennes functional [Ge]:

$$F = \int dx \left[ \frac{1}{N} [(\phi_A \ln \phi_A) + (\phi_B \ln \phi_B)] + \chi \phi_A \phi_B + \frac{a^2}{36 \phi_A \phi_B} |\nabla \phi|^2 \right] (x), \quad (7.6)$$

where  $N$  is the degree of polymerization,  $a$  the effective size of the monomer,  $\chi$  the Flory interaction parameter normalized by  $kT$ ,  $\phi_A$  and  $\phi_B$  the fractions of  $A$  and  $B$ ,  $\phi_A + \phi_B = 1$ . It differs from (7.5) in two main respects: the mixture is incompressible and the term accounting for the spatial variations contains a gradient square. If we impose in (7.6) the incompressibility constraint  $\rho_1 + \rho_2 = 1$  the first two terms reduce to the first two in (7.6) (apart from constants), and we remember that in the sharp interface limit the density becomes automatically constant, because the value of the density at the equilibrium is the same in the two phases. As far as the last term is concerned, it is well known [GL, GLP] that in models for alloys the mesoscopic conservative dynamics associated with a self-consistent interaction (van der Waals functional) or with the Ginsburg–Landau functional, involving instead a square gradient term, give the same qualitative sharp interface behaviour. The fact is that the two kinds of functionals have the same bulk terms but different terms to penalize nonhomogeneous profiles. This difference changes the interface properties but not the bulk properties which are relevant in the sharp interface limit. We find here the same situation: the behaviour of the two models in the sharp

interface limit is qualitatively the same, even if the constants such as the surface tension have different expressions. If we compute the second term in (7.5) on functions slowly varying on the scale where the interface is not sharp and Taylor expand in the width of the interface, then, to leading orders, we see a square gradient term. Nevertheless, the two models are different: let us just mention that the phase diagram for our system is based on three parameters because the density is not fixed and the critical parameter for the phase segregation is  $\beta\rho$ . This surely has consequences on the description of the early stage of the segregation process and of binodal and spinodal decomposition, but not on the late stages of the process.

Finally, we would briefly comment on the physical meaning of the second order equations. We write the equation for  $\psi^{(2)} = \bar{\mu}_1^{(2)} - \bar{\mu}_2^{(2)}$

$$\begin{aligned} \Delta_r \psi^{(2)}(r, t) &= \frac{S_1}{\hat{\rho}_1^{(0)}} - \frac{S_2}{\hat{\rho}_2^{(0)}} \quad r \in \Omega \setminus \Gamma_t, \\ \psi^{(2)}(r, t) &= \tau \bar{V} + H' \quad r \in \Gamma_t, \end{aligned} \quad (7.7)$$

$$V^{(1)} = \frac{T}{2[\bar{\rho}^+ - \bar{\rho}^-]} \left[ \frac{1}{\bar{\rho}} (\bar{\rho}^2 - |\bar{\varphi}|^2) [\bar{v} \cdot \nabla_r \psi^{(2)}]_{\pm}^{\pm} + \frac{2|\bar{\varphi}|}{\bar{\rho}} v \cdot \nabla_r \zeta^{(2)} \right] + Q' \quad r \in \Gamma_t,$$

where  $\zeta^{(2)}$  satisfies (6.48),  $\tau = (\beta/2|\bar{\varphi}|)[- \bar{\rho} + \int dz (w_1 + w_2)(z)]$  and  $Q', H'$  are known functions of the quantities  $\rho_i^{(1)}, d^{(1)}, \bar{\rho}_i$ . The boundary condition on  $\psi^{(2)}$  involves the velocity of the interface (which is known at this stage) and this is a typical non-equilibrium effect. In fact, the boundary condition in (7.1) owes its form to the fact that the fluid at the lowest order is in equilibrium and stays in equilibrium while the interface moves. On the other hand the motion of the interface does induce a modification of the equilibrium which is seen at next order with a term proportional to the velocity of the interface in the boundary condition. This is reminiscent of the so-called kinetic term in the modified Gibbs–Thomson condition in the model in [G], describing the growth of spherulites.

## Acknowledgments

RM is indebted to M C Carvalho and E Carlen for the enlightening discussions in the initial phase of this project and to R Esposito for many valuable discussions. RM would like to thank E Orlandi for kindly sending her the draft of the paper [CCO1] as well as for the interesting discussions.

## Appendix A. Uniqueness

In this appendix we prove that the equations for the  $\tilde{f}^{(n)}(z, r, t)$  we examined in section 5 have solutions whose dependence on the velocity is necessarily Gaussian. We will omit for simplicity the dependence on the other variables.

We consider the following set of equations for  $h_i(z, v), i = 1, 2, (z, v) \in \mathbb{R} \times \mathbb{R}^d$

$$v_z \partial_z h_i + F_i \partial_{v_z} h_i = L_\beta h_i, \quad (A.1)$$

where  $F_i = -\partial_z \int dz' U(|z - z'|) \rho_j(z') := -\partial_z V_i, i \neq j, \rho_i(z) = \int dv h_i(z, v)$ , with the conditions at infinity:

$$h_i(\pm\infty, v) = M(v) \rho_i^\pm \quad (A.2)$$

and show that it has only a solution of the form  $M(v) \rho_i(z)$ .

Put  $h_i = \psi_i(z, v) M(v) e^{-\beta V_i}$ .  $V_i$  is bounded due to the assumptions on  $U$ . Then,

$$v_z \partial_z \psi_i + F_i \partial_{v_z} \psi_i = \tilde{L}_i \psi_i, \quad (A.3)$$

where

$$\tilde{L}\psi_i = \frac{1}{M_\beta} \nabla_v \cdot (M_\beta \nabla_v \psi_i)$$

with the conditions at infinity

$$\psi_i(\pm\infty, v) = e^{+\beta V_i(\pm\infty)} \rho_i^\pm.$$

Multiply by  $M_\beta \psi_i$  and integrate over  $v$ :

$$\frac{1}{2} \partial_z (v_z \psi_i, \psi_i)_{M_\beta} + F_i \left( \psi_i, \frac{d}{dv_z} \psi_i \right)_{M_\beta} = -(\psi_i, \tilde{L}\psi_i)_{M_\beta}, \quad (\text{A.4})$$

where  $(h, g)_{M_\beta} = \int dv h(v)g(v)M_\beta(v)$ . We have

$$\frac{1}{2} \frac{d}{dz} (v_z \psi_i, \psi_i)_{M_\beta} - \frac{\beta}{2} F_i(v_z \psi_i, \psi_i)_{M_\beta} = -e^{-\beta V_i} (\nabla_v \psi_i, \nabla_v \psi_i)_{M_\beta} \quad (\text{A.5})$$

that we write as

$$\frac{d}{dz} [(v_z \psi_i, \psi_i)_{M_\beta} e^{-\beta V_i}] = -2(\nabla_v \psi_i, \nabla_v \psi_i)_{M_\beta} e^{\beta V_i}. \quad (\text{A.6})$$

We note that  $(v_z \psi_i, \psi_i)(\pm\infty) = 0$  because of the boundary conditions. Hence, by integrating over  $z$  we get

$$\int_{-\infty}^{+\infty} dz e^{-\beta V_i} \|\nabla_v \psi_i\|_\beta^2 = 0,$$

which implies  $\nabla_v \psi_i = 0$  a.e. since  $V_i$  is bounded, so that  $\psi_i = g(z)$ , a function only of the position. Going back to the original equation we see that  $g(z)$  has to be the front solution.

Next order equation.

We discuss now equation (6.19). A solution has been explicitly found as a Maxwellian times the density  $\tilde{\rho}^{(1)}$ . Suppose that there are two different solutions  $h$  and  $h'$  such that  $\rho_h = \rho_{h'}$ . Then, the equation for the difference is of the form investigated above, so that  $h - h' = 0$ . This means that there is a unique solution of the form  $M_\beta(v)\rho(z)$  in the class of solutions with fixed density  $\rho$ . Then, putting this expression back in the equation we determine  $\rho$ . The next order equations for  $\tilde{f}_i^{(n)}$  have a similar form, but the solutions are no longer of the form Maxwellian times a polynomial. The existence and uniqueness have to be proved by a different argument.

## Appendix B. Surface tension

The surface tension for a planar interface can be defined as the difference between the grand canonical free energy (pressure) of an equilibrium state with the interface and a homogeneous one [WR]. We can call this difference excess pressure. The pressure for this model is

$$\mathcal{P}(n_1, n_2) = \int dx p(n_1(x), n_2(x)),$$

$$p(n_1, n_2) = T(n_1 \log n_1 + n_2 \log n_2) + \frac{1}{2} n_1 U \star n_2 + \frac{1}{2} n_2 U \star n_1 - \mu_1 n_1 - \mu_2 n_2.$$

Consider the system in a cylinder of base  $(2L)^{d-1}$  and height  $M$  in the presence of a planar interface dividing the cylinder into two halves: in the upper half of the cylinder the densities are  $\rho_1^+, \rho_2^+$  and in the lower half of the cylinder densities are  $\rho_1^-, \rho_2^-$ , where  $\rho_1^\pm, \rho_2^\pm$  are the equilibrium values of the densities in the coexisting phases at temperature  $T$ . Then the excess pressure is given by [B]

$$\sigma = \lim_{L \rightarrow \infty} \frac{1}{(2L)^{d-1}} \lim_{M \rightarrow \infty} \int_{-L}^L dy_1 \dots \int_{-L}^L dy_{d-1} \int_{-M}^M dy_d [p(w_1, w_2) - p(\rho_1^+, \rho_2^-)],$$

where  $w_i(q)$  are the front solutions and smooth functions satisfying the equations

$$T \log w_i(q) + \int_{\mathbb{R}} dq' \tilde{U}(|q - q'|) w_j(q') = C_i, \quad (\text{B.1})$$

where  $\tilde{U}(q) = \int_{\mathbb{R}^2} dy U(\sqrt{q^2 + y^2})$  and  $C_i = \mu_i - T$  are constants determined by the conditions at infinity  $\rho_i^\pm$ . Note that  $f(\rho_1^+, \rho_2^-) = f(\rho_1^-, \rho_2^+)$  since  $\rho_1^\pm = \rho_2^\mp$  and that  $\mu_1 = \mu_2 = \mu$  in the coexisting region.

We rewrite the surface tension by using integration by part and the condition at infinity

$$\sigma = \int_{-\infty}^{+\infty} dz [p(w_1, w_2) - p(n_1^+, n_2^-)] = - \int_{-\infty}^{+\infty} dz z \frac{d}{dz} p(w_1(z), w_2(z)).$$

We have

$$\begin{aligned} \frac{d}{dz} p(w_1, w_2) &= T[(\log w_1 + 1)w_1' + (\log w_2 + 1)w_2'] + \frac{1}{2}[w_1' \tilde{U} \star w_2 \\ &\quad + w_2' \tilde{U} \star w_1 + w_1 \tilde{U} \star w_2' + w_2 \tilde{U} \star w_1'] - \mu(w_1' + w_2'). \end{aligned}$$

By using (B.1) and  $C_1 = C_2 = C$  we get

$$\begin{aligned} \frac{d}{dz} f(w_1, w_2) &= \frac{1}{2}[-w_1' \tilde{U} \star w_2 - w_2' \tilde{U} \star w_1 + w_1 \tilde{U} \star w_2' + w_2 \tilde{U} \star w_1'] \\ &\quad + (C + T)(w_1' + w_2') - \mu(w_1' + w_2') \end{aligned}$$

and for the surface tension, by remembering that  $C = \mu - T$ ,

$$-\frac{1}{2} \int dz dz' z \sum_{i \neq j} [-w_i'(z) \tilde{U}(z - z') w_j(z') + w_i(z) \tilde{U}(z - z') w_j'(z')].$$

In conclusion,

$$\sigma = \frac{1}{2} \int dz dz' (z - z') \sum_{i \neq j} [w_i'(z) \tilde{U}(z - z') w_j(z')].$$

### Appendix C. Forces

We show how to compute the terms  $\hat{g}_i^{(n)}$  and  $\bar{g}_i^{(n)}$  up to order 3. The procedure can be easily extended at any order.

For a slowly varying function  $h(r, t)$  we have that

$$\begin{aligned} U^\varepsilon \star h(r, t) &= \int_{\mathbb{R}^3} \varepsilon^{-3} U(\varepsilon^{-1}|r - r'|) h(r', t) dr' \\ &= \int_{\mathbb{R}^3} U(|x - x'|) [h(\varepsilon x', t) - h(\varepsilon x, t)] dx' + h(r, t) \int_{\mathbb{R}^3} U(|x - x'|) dx' \\ &= \int_{\mathbb{R}^3} U(|x - x'|) \left[ \varepsilon(x - x') \cdot \nabla_r h(r, t) \right. \\ &\quad \left. + \frac{\varepsilon^2}{2} \sum_{i,j} (x - x')_i (x - x')_j \frac{\partial^2}{\partial r_i \partial r_j} h(r, t) + \varepsilon^4 R_h(x, x') \right] dx' + h(r, t) \hat{U} \\ &= h(r, t) \hat{U} + \varepsilon^2 \Delta_r h(r, t) \bar{U} + \varepsilon^4 U \star R_h, \end{aligned} \quad (\text{C.1})$$

where  $\hat{U} = \int U(r) dr$ ,  $\bar{U} = \frac{1}{2} \int r^2 U(r) dr$ . We have used the isotropy of  $U$ . Hence we have

$$\hat{g}_i^{(n)} = \hat{U} \hat{\rho}_i^{(n)}, \quad n = 0, 1; \quad \hat{g}_i^{(2)} = \hat{U} \hat{\rho}_i^{(2)} + \bar{U} \Delta_r \hat{\rho}_i^{(0)}, \quad \hat{g}_i^{(3)} = \hat{U} \hat{\rho}_i^{(3)} + \bar{U} \Delta_r \hat{\rho}_i^{(1)}.$$

To compute the expansion of  $U^\varepsilon \star h$  for a fast varying function  $h(z, r, t)$  it is more convenient to use a local system of coordinates. For a given curve  $\Gamma$  and for any point  $s \in \Gamma$  we choose a reference frame centred in  $s$  with the axes 1, 2 along the directions of principal curvatures  $k_i$  and 3 in the direction of the normal. Consider two points  $r$  and  $r'$  and choose the reference frame centred in  $s(r) : r = s(r) + \varepsilon z\nu(r)$ . We denote by  $y_i$  and  $y'_i$  the microscopic coordinates of  $r$  and  $r'$  in this new frame. Then  $y_1 = y_2 = 0, y_3 = z$  and  $q'_i = \varepsilon^{-1}r'_i$ , the microscopic coordinates of  $r'$ , are related to  $y'_i$  by a linear transformation  $q'_i = A_{i\ell}y'_\ell$ . Moreover,  $z'$  is given in terms of  $y'_i$  by [GL]

$$\begin{aligned} z'(\{y'_i\}) &= y'_3 + \sum_{i=1,2} \frac{1}{2} [\varepsilon k_i y_i'^2 - 2\varepsilon^2 k_i^2 y_i'^2 y_3'] + \frac{1}{2} \varepsilon^3 \left( \sum_i k_i^2 y_i'^2 \right)^2 \\ &\quad - \varepsilon^3 \frac{1}{4} \left[ \sum_{ij} \partial_i^2 k_j y_i'^2 y_j'^2 (4k_i(1 - \delta_{ij}) + 3 - 2\delta_{ij}) \right] + O(\varepsilon^4), \end{aligned} \quad (\text{C.2})$$

where  $z' : r' = s(r') + \varepsilon\nu(r')z'$ .

We denote by  $\check{h}(y'_1, y'_2, y'_3, t)$  the function  $h(\{\varepsilon A_{i\ell}y'_\ell\}, z'(\{y'_i\}), t)$ . We have

$$\begin{aligned} (U^\varepsilon \star h)(z, r, t) &= \int_{\mathbb{R}^3} dy' U(|y - y'|) \check{h}(y'_1, y'_2, y'_3, t) \\ &= \int_{\mathbb{R}^3} dy' U(|y - y'|) \check{h}(0, 0, y'_3, t) + \frac{1}{2} \sum_{i=1,2} \left( \tilde{U}_{1,i} \star \frac{\partial^2 \check{h}}{\partial y_i'^2} \right)(z, r, t) \\ &\quad + \frac{1}{4} \sum_{i=1,2} \left( \tilde{U}_{2,i} \star \frac{\partial^4 \check{h}}{\partial y_i'^4} \right)(z, r, t) + \frac{1}{4} \sum_{i,j=1,2,i \neq j} \tilde{U}_{2,ij} \star \frac{\partial^4 \check{h}}{\partial y_i'^2 \partial y_j'^2} + \mathcal{Q}, \end{aligned}$$

where

$$\begin{aligned} \tilde{U}_{s,i}(|y_3 - y'_3|) &= \int_{\mathbb{R}^2} dy'_1 dy'_2 U \left( \sqrt{|y_3 - y'_3|^2 + |y'_1|^2 + |y'_2|^2} \right) |y'_i|^{2s} \\ \tilde{U}_{2,ij} &= \int_{\mathbb{R}^2} dy'_1 dy'_2 U \left( \sqrt{|y_3 - y'_3|^2 + |y'_1|^2 + |y'_2|^2} \right) |y'_i|^2 |y'_j|^2. \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial \check{h}}{\partial y'_i} &= \varepsilon \sum_j \bar{\nabla}_j h A_{ji} + \frac{\partial h}{\partial z} \frac{\partial z'}{\partial y'_i}, \\ \frac{\partial^2 \check{h}}{\partial y_k'^2} &= \varepsilon^2 \sum_{j\ell} A_{jk} A_{\ell k} \bar{\nabla}_j^2 h + \frac{\partial z'}{\partial y_k'} \left[ \varepsilon A_{jk} \bar{\nabla}_j \frac{\partial h}{\partial z} + \frac{\partial z'}{\partial y_k'} \frac{\partial^2 h}{\partial z^2} \right] + \frac{\partial h}{\partial z} \frac{\partial^2 z'}{\partial y_k'^2}. \end{aligned}$$

Using the relation between  $z'$  and  $y'_3$  (C.2) we see that the second term equals to

$$\frac{\partial h}{\partial z}(0, 0, y'_3, t) (\varepsilon k_k - \varepsilon^2 2k_k^2 y'_3).$$

It is true that [B]

$$\int_{\mathbb{R}^3} dy' U(|y - y'|) \frac{\partial h}{\partial z}(0, 0, y'_3, t) \sum_{i=1}^{d-1} \frac{k_i^2 y_i'^2}{2} = \frac{K}{2} \int_{\mathbb{R}} dz' (z' - z) \tilde{U}(|z' - z|) h(z', r, t). \quad (\text{C.3})$$

To compute the contributions at different orders in  $\varepsilon$  we go back to the specific curve  $\Gamma_t^\varepsilon$  and use the expansion  $d^\varepsilon(r, t) = \sum_n \varepsilon^n d^{(n)}(r, t)$  which implies  $k_i^\varepsilon = \sum_n \varepsilon^n k_i^{(n)}$  and  $A_{ij}^\varepsilon = \sum_n \varepsilon^n A_{ij}^{(n)}$ . In conclusion,

$$\begin{aligned} (U^\varepsilon \star h)(z, r) &= (\tilde{U} \star h)(z, r) + \varepsilon \frac{\bar{K}}{2} \int_{\mathbb{R}} dz' (z' - z) \tilde{U}(|z' - z|) h(z', r) \\ &\quad + \sum_{i=1,2} \left[ \varepsilon^2 \left( \tilde{U}_{1,i} \star D_{1,i}(h) + \tilde{U}_{2,i} \star D_{2,i}(h) \right. \right. \\ &\quad \left. \left. + \sum_{j \neq i} \tilde{U}_{2,ij} \star D_{2,ij}(h) + \frac{K^{(1)}}{2} C(h) \right) + \varepsilon^3 B_3 \right] \\ &:= (\tilde{U} \star h)(z, r) + \sum_{n=1}^3 \varepsilon^n B_n(h) + \tilde{R}_h, \end{aligned} \quad (\text{C.4})$$

where  $\tilde{R}_h$  is of order  $\varepsilon^4$  and

$$\begin{aligned} D_{1,i}(h) &= \frac{1}{2} \sum_{j\ell} A_{ji}^{(0)} A_{\ell i}^{(0)} \bar{\nabla}_{j\ell}^2 h - \frac{\partial h}{\partial z} 2(k_i^{(0)})^2 y_3', & D_{2,i}(h) &= \frac{3 - 6y_3'}{4} (k_i^{(0)})^2 \frac{\partial^2 h}{\partial z^2}, \\ C(h) &= \int_{\mathbb{R}} dz' (z' - z) \tilde{U}(|z' - z|) h(z', r), & D_{2,ij}(h) &= \frac{1}{4} k_i^{(0)} k_j^{(0)} \frac{\partial^2 h}{\partial z^2}. \end{aligned}$$

We do not write explicitly the long and uninteresting formula for  $B_3$ . Hence we have

$$\begin{aligned} \tilde{g}_i^{(0)} &= \tilde{U} \star \tilde{\rho}_i^{(0)}, \\ \tilde{g}_i^{(1)} &= \tilde{U} \star \tilde{\rho}_i^{(1)} + B_1(\tilde{\rho}_i^{(0)})(z, r), \\ \tilde{g}_i^{(2)} &= \tilde{U} \star \tilde{\rho}_i^{(2)} + B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)}), \\ \tilde{g}_i^{(3)} &= \tilde{U} \star \tilde{\rho}_i^{(3)} + B_1(\tilde{\rho}_i^{(2)}) + B_2(\tilde{\rho}_i^{(1)}) + B_3(\tilde{\rho}_i^{(0)}). \end{aligned} \quad (\text{C.5})$$

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