DIFFUSIVE LIMIT OF ASYMMETRIC SIMPLE EXCLUSION: THE NAVIER-STOKES CORRECTION

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PROBLEM AND RESULTS.

We consider the simple exclusion process (SEP) on a cubic sublattice $\Lambda_L \subset \mathcal{Z}^d$ of size $2L+1$, with periodic boundary conditions. The dynamics is the following: let e denote one of the $2d$ possible directions in $\mathcal{Z}^d$. A particle in the site x, independently of the others, waits for an exponential time and jumps, with probability proportional to $p_e \geq 0$, to the site $x+e$, if it is empty; otherwise stays in x and the process starts again. We denote by $\eta_x(\tau) = 0, 1$ the number of particles in the site x at time τ and by $\mathcal{L}$ the generator of the process: $\mathcal{L}f = \sum_b \mathcal{L}_b f$, the sum running on the set of all oriented bonds $b = (x, y)$ in $\mathcal{Z}^d$ such that $y - x = e$ and

$$\mathcal{L}_b f = p_e \eta_x [f(\eta^b) - f(\eta)] \tag{1}$$

with

$$(\eta^b)_z = (\eta^{x,y})_z = \begin{cases} \eta_y, & z = x \\ \eta_x, & z = y \\ \eta_z, & \text{otherwise.} \end{cases}$$

It is convenient to choose the normalization $p_{-e} + p_e = 2$ for all e.

In the case $p_e = p_{-e}$ for all e (symmetric simple exclusion process) the hydrodynamical limit is non trivial only in the diffusive scaling, $z = \varepsilon x$, $t = \varepsilon^2 \tau$, where x and τ are the microscopic space and time, while z and t are the macroscopic ones, and $\varepsilon = L^{-1}$. Under this scaling, as $\varepsilon \to 0$ the limiting equation reduces to the diffusion equation with diffusion coefficient 1. The case $p_e \neq p_{-e}$, called asymmetric simple exclusion process (ASEP), is much more difficult.

Define the product measure in Λ_L, with chemical potential β, as

$$d\mu_{\beta,L} = Z_\beta^{-1} \exp[\beta \sum_{x \in \Lambda_L} \eta_x], \quad \eta_x \in \{0, 1\}, \tag{2}$$

Z_β being the normalization. $\mu_{\beta,L}$ is invariant for the dynamics because of the periodic b.c., but the dynamics is not reversible with respect to it. The behavior of the system is non trivial already in the hyperbolic (Euler) scaling ($\tau = \varepsilon^{-1} t$) and is given by the non viscous Burgers equation (see ref.[1])

$$\partial_t \rho + \delta \cdot \nabla_z [\rho(1-\rho)] = 0, \tag{3}$$

$\delta_e = p_e - p_{-e}$. Since the system moves already on the Euler scale, it is not clear the meaning of the diffusive limit for this system. For weak asymmetries, namely for the case δ_e of order ε, in the diffusive scaling the process converges to a diffusion with a non linear drift [2,3] . However this case is a small perturbation of the symmetric one and does not manifest the asymmetric nature of the problem. In fact the diffusive constant so obtained is still 1.

We consider[4] the ASEP with δ of order 1, but in a restricted class of initial data for which there is a different mechanism to get a well defined diffusive limiting behavior. This is suggested by the analogy with the study of the incompressible regime for a fluid[5]. Suppose that the initial density $\rho_0(\varepsilon x)$ is of the form

$$\rho_0(\varepsilon x) = \theta - \varepsilon u_0(\varepsilon x)$$

with $\theta \in (0,1)$. Then, assuming that this form of the density persists at time t, the macroscopic current $J = \delta[\rho(1-\rho)]$ becomes

$$J = \delta\theta(1-\theta) - \varepsilon v u - \varepsilon^2 \delta u^2, \quad v = (1-2\theta)\delta.$$

The equation for the field u is expected to be

$$\partial_t u + \varepsilon^{-1} v \cdot \nabla_z u + \delta \cdot \nabla_z u^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 u}{\partial z_i \partial z_j} \tag{4}$$

with $D_{i,j}$ some diffusion matrix. A change of reference frame

$$m(z,t) = u(z + \varepsilon^{-1} v t, t)$$

allows to remove the diverging term and the equation for $m(z,t)$ is the viscous Burgers equation:

$$\partial_t m + \delta \cdot \nabla_z m^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 m}{\partial z_i \partial z_j} \tag{5}$$

This suggests us to assume an initial measure with density w.r.t $\mu_{beta,L}$ given by

$$\psi_0 = Z_\lambda^{-1} \exp\{\varepsilon \sum_x \lambda(\varepsilon x) \eta_x\} \tag{6}$$

for some smooth function $\lambda(z)$, Z_λ being the normalization. Since the product measure with constant chemical potential is invariant, it is natural to think of $\psi_0 d\mu_{\beta,L}$ as a *local equilibrium* for the system.

The average occupation number in the initial local equilibrium state is, up to terms of higher order in ε, $\theta - \varepsilon m_0(\varepsilon x)$, with $\theta = e^\beta/(1+e^\beta)$ and $m(z) = -\theta(1-\theta)\lambda(z)$. By above heuristic arguments one expects that, starting with this initial datum, the occupation number at time t, in the moving frame, differs from θ by $\varepsilon m(z,t)$ with $m(z,t)$ solving (5) with initial datum $m_0(z)$. To state our result we introduce the empirical measure

$$\nu_\epsilon(z,t) = \epsilon^{d-1} \sum_x \delta(z - \epsilon x)(\theta - \eta_x(\varepsilon^{-2} t)).$$

and $\hat{\nu}_\epsilon(z,t) = \nu_\epsilon(z - \epsilon^{-1} v t, t)$. Next theorem states the weak convergence of $\hat{\nu}_\epsilon(z,t)$ to $m(z,t)$, if $d \geq 3$.

THEOREM 1.1 *For $d \geq 3$, $\hat{\nu}_\varepsilon(z,t)$ converges weakly in probability to $m(z,t)$ solving the equation (5). The diffusion matrix satisfies $1 < D < \infty$ (as a matrix) and it is formally given by the Green-Kubo formula.*

Renarks.

1. The diffusion matrix is expected to be infinite in dimension 1 and 2 [1]. Actually our method relies on a multiscale analysis which fails in dimension less than 3.
2. The fact that $D > 1$ is, in our opinion, related to the presence of a Navier-Stokes correction in this model. In fact, we interpret the non reversibility of the generator $\mathcal{L}$ as due to a *deterministic* motion added to the stochastic motion due to the symmetric part of the generator. The last one produces a diffusion process with diffusion coefficient 1, hence there is a non vanishing contribution to the diffusion due to the antisymmetric part of the generator.
3. The Green-Kubo formula does not make really sense, hence we give a variational formula for D which is a way to give a rigorous sense to the Green-kubo formula.
4. We cannot expect a isotropic diffusion because of the presence of the cubic lattice which interferes in some complicated way with the driving field δ.

Theorem 1.1 is an immediate consequence of the fact that we can construct a local equilibrium measure which approximates, in a suitable sense, up to order ε^2, the true distribution density.

Let f_t be the distribution density at time t with respect to $\mu_{\beta,L}$. Then f_t satisfies the rescaled forward equation

$$\partial_t f_t = \varepsilon^{-2}\mathcal{L}^* f_t$$

Here $\mathcal{L}^*$, defined by

$$\mathcal{L}^* f = \sum_b \mathcal{L}_b^* f, \qquad \mathcal{L}_b^* f = p_{-e}\eta_x[f(\eta^b) - f(\eta)],$$

is the adjoint of $\mathcal{L}$ in $L_2(d\mu_{\beta,L})$ with periodic b.c. The distribution density is initially $f_0 = \psi_0$.

Recall that the relative entropy of the densities f and g (w.r.t. $d\mu_\beta$) is defined by

$$s(f|g) = \varepsilon^d \int f \log \frac{f}{g} d\mu_\beta \tag{7}$$

and put $s(f) \equiv s(f|1)$.

Note that $s(f_0) = O(\varepsilon^2)$, so that $s(f_t) = O(\varepsilon^2)$ too because $s(f_t)$ is constant in time. A better approximation is provided by a local equilibrium ψ_t with the right choiche of the chemical potential. In fact, let

$$\psi_t = \frac{1}{Z(t)} \exp\{\varepsilon \sum_x \lambda(\varepsilon x - \varepsilon^{-1} vt, t)\eta_x\}, \tag{8}$$

where $Z(t)$ is the normalization (w.r.t. $d\mu_\beta$) and λ satisfies the equation

$$\frac{\partial}{\partial t}\lambda(z,t) - \theta(1-\theta)\sum_{e>0} \delta_e(\partial_e \lambda^2)(z,t) = \sum_{e>0}\sum_{e'>0} D_{ee'}(\partial_e \partial_{e'}\lambda)(z,t). \tag{9}$$

Here $\partial_e \lambda(z) = e \cdot \nabla$