

Stability of the Front under a Vlasov-Fokker-Planck Dynamics

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Abstract

We consider a kinetic model for a system of two species of particles interacting through a long range repulsive potential and a reservoir at given temperature. The model is described by a set of two coupled Vlasov-Fokker-Planck equations. The important front solution, which represents the phase boundary, is a stationary solution on the real line with given asymptotic values at infinity. We prove the asymptotic stability of the front for small symmetric perturbations.

Keywords: *Vlasov, Fokker-Planck, phase transition, stability, fronts*

1. Introduction and Notations.

The dynamical study of phase transitions has been tackled, among the others, with an approach based on kinetic equations modeling short range and long range interactions which are responsible of critical behaviors. An example of such models has been proposed in [2] where the authors study a system of two species of particles undergoing collisions regardless of the species and interacting via long range repulsive forces between different species. A simplification of such model has been considered in [15] where a kinetic model has been introduced for a system of two species of particles interacting through a long range repulsive potential and with a reservoir at a given temperature T . The interaction with the reservoir is modeled by a Fokker-Planck operator and the interaction between the two species by a Vlasov force. The system is described by the one-particle distribution functions $f_i(x, v, t)$, $i = 1, 2$, with $(x, v) \in \Omega \times \mathbb{R}^3$ the position and velocity of the particles. The distribution functions f_i are solutions of a system of two coupled Vlasov-Fokker-Planck (VFP) equations in a domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t f_i + v \cdot \nabla_x f_i + F_i \cdot \nabla_v f_i = L f_i, \quad (1.1)$$

where

$$L f_i = \nabla_v \cdot \left(M \nabla_v \left(\frac{f_i}{M} \right) \right), \quad (1.2)$$

and M is the Maxwellian

$$M = \left(\frac{\beta}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{\beta}{2} v^2}$$

with mean zero and variance $\beta^{-1} = T$ which is interpreted as the temperature of the thermal reservoir. The self-consistent Vlasov force, representing the repulsion between particles of different species, is

$$F_i = -\nabla_x \int_{\Omega} dx' U(|x - x'|) \int_{\mathbb{R}^3} dv f_j(x', v, t).$$

Here and in the rest of the paper we use the convention that the indices i and j are related as follows: if $i = 1$ then $j = 2$, while, when $i = 2$ then $j = 1$. The potential function U is a positive, bounded, smooth, monotone decreasing function on \mathbb{R}_+ , with compact support and $\int_{\mathbb{R}^3} dx U(|x|) = 1$. There is a natural Liapunov functional, the *free energy*, for this dynamics,

$$\begin{aligned} \mathcal{G}(f_1, f_2) := & \int_{\Omega \times \mathbb{R}^3} dx dv \left[(f_1 \ln f_1) + (f_2 \ln f_2) + \frac{\beta}{2} (f_1 + f_2) v^2 \right] \\ & + \beta \int_{\Omega \times \Omega} dx dy U(|x - y|) \int_{\mathbb{R}^3} dv f_1(x, v) \int_{\mathbb{R}^3} dv' f_2(y, v'). \end{aligned}$$

In fact, we have that

$$\frac{d}{dt} \mathcal{G}(f_1, f_2) = - \sum_{i=1,2} \int_{\Omega \times \mathbb{R}^3} dx dv \frac{M^2}{f_i} \left| \nabla_v \frac{f_i}{M} \right|^2 \leq 0$$

and the time derivative is zero if and only if f_i are of the form $f_i = \rho_i M$, where ρ_i are functions only of the position. If we put these expressions back in the VFP equations we see that the stationary solutions of (1.1) have densities satisfying the equations

$$\ln \rho_i(x) + \beta \int_{\Omega} dx' U(|x - x'|) \rho_j(x') = C_i, \quad x \in \Omega, \quad i = 1, 2 \quad (1.3)$$

and C_i are arbitrary constants, related to the total masses $n_i |\Omega|$ of the components of the mixture. Moreover, replacing f_i by $\rho_i M$ in the functional \mathcal{G} and integrating out the velocity variable we obtain a functional of the densities ρ_i

$$\mathcal{F}(\rho_1, \rho_2) = \int_{\Omega} dx (\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2) + \beta \int_{\Omega \times \Omega} dx dy U(|x - y|) \rho_1(x) \rho_2(y) \quad (1.4)$$

The Euler-Lagrange equations for the minimization of \mathcal{F} with the constraint on the total masses

$$\int_{\Omega} dx \rho_i(x) = n_i |\Omega|, \quad i = 1, 2, \quad (1.5)$$

are exactly (1.3). We set $n = n_1 + n_2$, the total average density.

In [4] it is proved that for $n\beta \leq 2$, equations (1.3) in a torus have a unique homogeneous solution, while for $n\beta > 2$ there are non homogeneous solutions. To explain the physical meaning of these non homogeneous solutions, we write the functional $\mathcal{F}(\rho_1, \rho_2)$ in the following equivalent form

$$\mathcal{F}(\rho_1, \rho_2) = \int_{\Omega} dx f(\rho_1, \rho_2) + \frac{\beta}{2} \int_{\Omega^2} dx dy U(|x-y|) [\rho_1(x) - \rho_1(y)] [\rho_2(y) - \rho_2(x)]$$

where $f(\rho_1, \rho_2)$ is the thermodynamic free energy made of the entropy and the internal energy:

$$f(\rho_1, \rho_2) = \rho_1 \log \rho_1 + \rho_2 \log \rho_2 + \beta \rho_1 \rho_2$$

The function $f(\rho_1, \rho_2)$ is not convex and has, for any given temperature $T = \beta^{-1}$, two symmetric (under the exchange $1 \rightarrow 2$) minimizers if the total mass $n = \frac{1}{|\Omega|} \int_{\Omega} dx (\rho_1 + \rho_2)$ is larger than a critical value $(2T)^{-1}$. Indeed, there are two positive numbers $\rho^+ > \rho^- > 0$ such that one minimizer is given by $\rho_1 = \rho^+, \rho_2 = \rho^-$ and the other by exchanging the indices 1 and 2. In other words, this system undergoes a first order phase transition with coexistence of two phases, one richer in the presence of species 1 and the other richer in the presence of species 2. In the rest of the paper we assume that the parameters T and $n = \rho^+ + \rho^-$ are in the phase transition region where $\frac{n}{T} > 2$. If we look for the minimizers of the free energy functional \mathcal{F} under the constraints (1.5) the only minimizers are homogeneous if we fix (n_1, n_2) equal to one of the two minimizers of $f(\rho_1, \rho_2)$.

Otherwise, if $n_i \in (\rho^-, \rho^+)$, $i = 1, 2$, below the critical value non homogeneous profiles may have lower free energy. The structure of the minimizing profiles of density will be as close as possible to one of the two minimizers of f : they will be close to one of the minimizing values in a region B , close to the other minimizing value in the complement but for a separating region called *interface* where the minimizing profiles will interpolate smoothly between the two values. A precise statement of this is proved in [4] under the assumption that the size of Ω is large compared to the range of the potential U .

We can conclude then that the minimizers of \mathcal{G} in a torus will be Maxwellians times densities ρ_i of the form discussed above. Since \mathcal{G} is a Liapunov functional, we expect that the minimizers are related to the stable solutions of the equations. In this paper we want to study the stability of the non homogeneous stationary solutions of the equations (1.1), which are minimizers of the kinetic free energy \mathcal{G} .

Since planar interfaces play an important role in the study of the evolution of general interfaces, in this paper we focus on the so called *front* solutions, i.e. one-dimensional infinite volume solutions, with

$$x = (0, 0, z) \quad -\infty < z < \infty .$$

The reason for choosing this setup is that in such a situation we know many more properties of the minimizers.

To be more precise, we introduce the *excess free energy functional* in one dimension on the infinite line defined as

$$\hat{\mathcal{F}}(\rho_1, \rho_2) := \lim_{N \rightarrow \infty} [\mathcal{F}_N(\rho_1, \rho_2) - \mathcal{F}_N(\rho^+, \rho^-)] \quad (1.6)$$

where \mathcal{F}_N is the free energy associated to the interval $[-N, N]$ and (ρ^+, ρ^-) is a homogeneous minimizer of the thermodynamic free energy f . We note that $\mathcal{F}_N(\rho^+, \rho^-) = \mathcal{F}_N(\rho^-, \rho^+)$. We look for the minimizers of the excess free energy such that $\lim_{z \rightarrow \pm\infty} \rho_1(z) = \rho^\pm$, $\lim_{z \rightarrow \pm\infty} \rho_2(z) = \rho^\mp$, because otherwise the limit defining $\hat{\mathcal{F}}$ would not be finite. By the translation invariance of $\hat{\mathcal{F}}$ the minimizers are degenerate. We remove the degeneration by imposing the *centering condition*, $\rho_1(0) = \rho_2(0)$. In [5] it is proved that

Theorem 1.1. *Suppose that the total density $n = \rho^+ + \rho^-$ and the inverse temperature $\beta = T^{-1}$ are in the phase transition region $\beta n > 2$. Then there exists a unique C^∞ positive minimizer (front) $w = (w_1(z), w_2(z))$, with $w_1(z) = w_2(-z)$, for the one-dimensional excess free energy $\hat{\mathcal{F}}$, defined in (1.6), in the class of continuous functions $\rho = (\rho_1, \rho_2)$ such that*

$$\lim_{z \rightarrow \pm\infty} \rho_1 = \rho^\pm, \quad \lim_{z \rightarrow \pm\infty} \rho_2 = \rho^\mp.$$

The properties of the minimizer are: w_1 is monotone increasing and w_2 is monotone decreasing and

$$\rho^- < w_i(z) < \rho^+$$

for any $z \in R$.

Moreover, the front w is smooth and satisfies the Euler-Lagrange equations (1.3); its derivatives w' satisfy the equations

$$\frac{w'_1(z)}{w_1(z)} + \beta(U * w'_2)(z) = 0, \quad \frac{w'_2(z)}{w_2(z)} + \beta(U * w'_1)(z) = 0 \quad (1.7)$$

The front w converges to its asymptotic values exponentially fast, in the sense that there is $\alpha > 0$ such that

$$|w_1(z) - \rho_\mp| e^{\alpha|z|} \rightarrow 0 \text{ as } z \rightarrow \mp\infty, \quad |w_2(z) - \rho_\pm| e^{\alpha|z|} \rightarrow 0 \text{ as } z \rightarrow \mp\infty.$$

The functions w_i have derivatives of any order which vanish at infinity exponentially fast.

Our main result is the stability of these fronts for the VFP dynamics, under suitable assumptions on the initial data. To state the result, we write f_i , solutions of (1.1), as

$$f_i = w_i M + h_i .$$

Then, the perturbation h_i satisfies

$$\partial_t h_i + G_i h_i = L h_i - F_i(h) \partial_{v_z} h_i, \quad (1.8)$$

where the operators G_i are defined by

$$G_i h_i = v_z \partial_z h_i - (U * w'_j) \partial_{v_z} h_i + \left(U * \partial_z \int_{\mathbb{R}^3} dv h_j(\cdot, v, t) \right) \beta v_z M w_i \quad (1.9)$$

while the force $F_i(h)$ due to the perturbation is

$$F_i(h) = -\partial_z \int_{\mathbb{R}} dz' U(z - z') \int_{\mathbb{R}^3} dv h_j(z', v, t). \quad (1.10)$$

We define (\cdot, \cdot) as the L^2 inner product for two scalar functions (on \mathbb{R} or $\mathbb{R} \times \mathbb{R}^3$ depending on the context), while $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product for vector-valued functions, and we denote $\|\cdot\|$ as their corresponding L^2 norms. Furthermore, we define the weighted L^2 inner products as

$$(f_i, g_i)_M = \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{w_i M} f_i g_i, \quad \langle f, g \rangle_M = \sum_{i=1,2} \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{w_i M} f_i g_i,$$

with corresponding weighted L^2 norms by $\|\cdot\|_M$. We also define the dissipation rate as

$$\|g\|_D^2 = \|(I - P)g\|_M^2 + \|\nabla_v(I - P)g\|_M^2, \quad (1.11)$$

where P is the L^2 projection on the null space of L given by $\{cM, c \in \mathbb{R}^2\}$, for any given t, z . We also define the γ -weighted norms as

$$\|g\|_{M,\gamma} = \|z_\gamma g\|_M \quad \|g\|_{D,\gamma} = \|z_\gamma g\|_D,$$

with

$$z_\gamma = (1 + |z|^2)^\gamma.$$

In the following we will also denote by ∂h the couple of derivatives $\{\partial_t h, \partial_z h\}$ with the abuse of notation $\|\partial h\|^2 = \|\partial_z h\|^2 + \|\partial_t h\|^2$ for any of the norms appearing below. When there is no risk of ambiguity, to make the notation shorter, for any two vectors $f = (f_1, f_2)$ and $g = (g_1, g_2)$ we will denote by fg the vector with components $(f_1 g_1, f_2 g_2)$.

The following theorem will be proved in Section 4:

Theorem 1.2. *We assume that $h = (h_1, h_2)$ at time zero has the following symmetry property in (z, v)*

$$h_1(z, v, 0) = h_2(-z, Rv, 0), \quad Rv = (v_x, v_y, -v_z). \quad (1.12)$$

There is δ_0 small enough such that:

1. If $\|h(0)\|_M + \|\partial h(0)\|_M \leq \delta_0$, then there is a unique global solution to (1.8) such that for some $K > 0$

$$\begin{aligned} & \frac{d}{dt} \left(K(\|h(t)\|_M^2 + \|\partial_t h(t)\|_M^2) + \|\partial_z h(t)\|_M^2 \right) \\ & + K\nu_0(\|h(t)\|_D^2 + \|\partial_t h(t)\|_D^2) + \nu_0\|\partial_z h(t)\|_D^2 \leq 0. \end{aligned} \quad (1.13)$$

2. If, for $\gamma > 0$ sufficiently small,

$$\|h(0)\|_{M,\gamma} + \|\partial h(0)\|_{M,\frac{1}{2}+\gamma} \leq \delta_0$$

then there is constant $C > 0$ such that

$$\sup_{0 \leq t \leq \infty} \|h(t)\|_{M,\gamma} + \sup_{0 \leq t \leq \infty} \|\partial h(t)\|_{M,\frac{1}{2}+\gamma} \leq C(\|h(0)\|_{M,\gamma} + \|\partial h(0)\|_{M,\frac{1}{2}+\gamma}). \quad (1.14)$$

Moreover, we have the decay estimate

$$\|h(t)\|_M^2 + \|\partial h(t)\|_M^2 \leq C \left[1 + \frac{t}{2\gamma}\right]^{-2\gamma} \left[\|h(0)\|_{M,\gamma}^2 + \|\partial h(0)\|_{M,\frac{1}{2}+\gamma}^2\right]. \quad (1.15)$$

A key remark to prove Theorem 1.2 is that, since the equation preserves the symmetry property (1.12), the perturbations $h_i(z, v, t)$ have the same symmetry property (1.12) at any time. The proof of the theorem is based on energy estimates and takes advantage of the fact that at time zero the perturbation is small in a norm involving also the space and the time derivatives. To close the energy estimates, we use the spectral gap for the Fokker-Planck operator L to control $(I - P)h$, the part of h orthogonal to the null space of L , and the conservation laws to control Ph , the component of h in the null space of L , in terms of $(I - P)h$, like the method used in [9].

The main difficulty in our context is the control of the hydrodynamic part Ph (which can be written as $Ph = a_h M$ for some $a_h(z, t) \in \mathbb{R}^2$), in the presence of the Vlasov force with large amplitude. Because of the Vlasov force, the hydrodynamic equations do not give directly the control of the norm of Ph but instead of a norm involving the operator A , the second variation of the free energy $\hat{\mathcal{F}}$ at the front w , which is given, for any $g = (g_1, g_2)$ by

$$\langle g, Ag \rangle := \sum_{i=1}^2 \int_{\mathbb{R}} dz g_i(z) (Ag)_i(z) = \frac{d^2}{ds^2} \hat{\mathcal{F}}(w + sg) \Big|_{s=0}.$$

The action of the operator A on g is

$$(Ag)_1 = \frac{g_1}{w_1} + \beta U * g_2, \quad (Ag)_2 = \frac{g_2}{w_2} + \beta U * g_1. \quad (1.16)$$

Since w is a minimizer of $\hat{\mathcal{F}}$ the quadratic form on the left hand side is non negative and the vanishing of the first variation of $\hat{\mathcal{F}}$ gives the Euler-Lagrange equations

$$\frac{\delta \hat{\mathcal{F}}}{\delta \rho_i}(w) = \log w_i + \beta U * w_j - C_i = 0, \quad i = 1, 2.$$

Differentiating with respect to z and using the prime to denote the derivative with respect to the z variable, it results

$$(Aw')_i = \frac{w'_1}{w_1} + \beta U * w'_j = 0 ,$$

which shows that w' is in the null space of A . Indeed, one can show (see Section 2) that w' spans the null space of A and that there exists a constant $\lambda > 0$ (spectral gap) such that

$$\langle g, Ag \rangle \geq \lambda \sum_{i=1}^2 \int_{\mathbb{R}} dz \frac{1}{w_i} |(I - \mathcal{P})g_i|^2$$

where \mathcal{P} is the projector on the null space of A .

Hence, by estimating suitable norms of Aa_n , by means of the spectral gap for A we gain control on the component of Ph in the orthogonal to the null space of A . The component of Ph in the null space of A is still uncontrolled by this method. Let us write $a_n = \alpha w' + (I - \mathcal{P})a_n$. What is missing at this stage is an estimate for $\alpha(t) = \langle a_n(\cdot, t), w' \rangle$ for large times. We would like to show that $\alpha(t)$ vanishes asymptotically in time, which amounts to prove that the solution of the Vlasov-Fokker-Plank equations (VFP) converges to the initial front. The existence of a Liapunov functional for this dynamics forces the system to relax to one of the stationary points for the functional, which are of the form Mw^x , with w^x any translate by x of the symmetric front w . Then, it is the conservation law, in the form

$$\int_{\mathbb{R} \times \mathbb{R}^3} dz dv [f(z, v, t) - M(v)w(z)] = 0$$

which should select the front the solution has to converge to. But this is a condition requiring the control of the L^1 norm of the solution while our energy estimates control some weighted L^2 norm. In the approach in [9] the conservation law is used in problems in finite domains or in infinite domains but in dimension greater or equal than 3. The problem we are facing here is analogous to the one in [7] and we refer to it for more discussion. One can realize the connections between the problem discussed here and the one in [7] by looking at the hydrodynamic limit of the model. In [15] it is proved that the diffusive limit of the VFP dynamics is

$$\partial_t \bar{\rho} = \nabla \cdot \left(\mathcal{M} \nabla \frac{\delta \mathcal{F}}{\delta \bar{\rho}} \right), \quad \mathcal{M} = \beta^{-1} \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \quad (1.17)$$

where $\bar{\rho} = (\rho_1, \rho_2)$, $\frac{\delta \mathcal{F}}{\delta \bar{\rho}}$ denotes the functional derivative of \mathcal{F} with respect to $\bar{\rho} = (\rho_1, \rho_2)$ and \mathcal{M} is the 2×2 mobility matrix. These equations are

in the form of a gradient flow for the free energy functional as the equation considered in [7], which is an equation for a bounded magnetization $m(x, t) \in [-1, 1]$:

$$\partial_t m = \nabla \cdot \left[\sigma(m) \frac{\delta \mathcal{F}}{\delta m} \right]$$

where $\sigma(m) = \beta(1 - m^2)$ and \mathcal{F} is a suitable non local free energy functional. In [7] the stability result is obtained by using suitable weighted L^2 norms, with a weight $|x|$, which allow to control the tails of the distribution and hence a control of the L^1 norm. This is possible essentially because the equation is of diffusive type.

Unfortunately, we cannot use directly the approach in [7] since the dissipation in the kinetic model is given by the Fokker-Plank operator and does not produce directly diffusion on the space variable. In fact, we are able to use, as explained above, γ -weighted norms (in space) with a weight z_γ , with γ small, which are not enough to control the L^1 norm. Hence, to overcome the difficulty, we consider a special initial datum. We assume, as explained before, that h at initial time has the particular symmetry property (1.12). It is easy to see that this property is conserved by the dynamics so that h is symmetric at any later time. We note that also wM is symmetric while w' is antisymmetric in the z variable. This implies the vanishing at time $t = 0$ of $\sum_{i=1}^2 \int_{\mathbb{R} \times \mathbb{R}^3} dz dv h_i(z, v, t) M(v) w'_i(z)$, the component of a on the null space of A , which consequently is zero at any later time.

The symmetry assumption is crucial in the proof of Lemma 3.1 and Theorem 3.2. It is used in the proofs to establish that $\langle a, w' \rangle$ and $\langle a', w'' \rangle$ vanish. It is worth to notice that these are the only points of the argument depending on the symmetry condition and the rest of the argument would still work if the symmetry condition were replaced by a suitable estimate on $\langle a, w' \rangle$ and $\langle a', w'' \rangle$.

Even with the symmetry assumption (1.12), the estimate for the hydrodynamic part Ph is delicate. Based on the precise spectral information of A , we need to further study the derivative of A , $(Ag)' = \frac{\partial}{\partial z} Ag$. To this end, we employ the decomposition (2.6) for *each* component of g and a contradiction argument to establish an important lower bound for $(Ag)'$, (Theorem 2.4). Furthermore, in order to get the time decay rate, we use the additional polynomial weight function z_γ and a trick of interpolation to carefully derive the corresponding energy estimate in a bootstrap fashion. Once again, Theorem 2.4 and its corollary (Lemma 2.5) are crucial to control local L^2 norm of Ph in terms of its z -derivative.

It is worth to stress that our result does not rely on a smallness assumption on the potential, like for example in [18], where it is proved the stability in L^1 of the constant stationary state for a one component VFP equation, on a torus, for general initial data. The assumption of small U in [18] guarantees the uniqueness of the stationary state, namely it means not to be in the phase transition region. On the contrary, we are working with values of the parameters (temperature and asymptotic values of densities, ρ^\pm) in the

phase transition region. For values of the parameters $\rho^+ = \rho^- = \frac{n}{2}$, $\beta n \leq 2$ the minimizer is unique and we can prove that the constant solution (with $w_i(z) = \frac{n}{2}$) is stable, by a simplified version of the proof given here. The critical value $\beta n = 2$ is selected by the fact that the analogous of the operator A , that comes out from the linearization around the constant solution, is positive and has spectral gap for $\beta n < 2$ (it coincides with the operator called L_0 in Theorem 2.2). We expect also that the constant solution will become unstable above this critical value.

Finally, we want to return to the kinetic model proposed in [2], mentioned at the beginning of this section and studied in a series of papers [3], in which the Fokker-Planck term is replaced by a Boltzmann kernel to model species-blind collisions between the particles. The dynamics is described by a set of two Vlasov-Boltzmann equations, coupled through the Boltzmann collisions and the Vlasov terms and conserve not only the total masses but also energy and momentum. The stationary solutions are the same as in the previous model, Maxwellians times densities ρ_i satisfying (1.3), so that one could study the stability of these solutions with respect to the Vlasov-Boltzmann dynamics. This result is more difficult to get due to the non linearity of the Boltzmann terms. The first results on the stability of the Maxwellian are proved in [16], [14]. Recently, it has been proved by energy methods in a finite domain or in \mathbb{R}^3 in [9] ([11] for soft potentials) who has also extended the method to cover other models involving self-consistent forces and singular potentials [10] and in \mathbb{R}^d in [13], who also proved the stability of a $1 - d$ shock. The stability of the non homogeneous solution for a Boltzmann equation with a given small potential force has been proved in [17]. We are not aware of analogous results for non small force, but a very recent one, [1] relying on the assumption that the potential is compactly supported in \mathbb{R}^3 . Our method is in principle suited to prove stability under Vlasov-Boltzmann dynamics on a finite interval, but what is still lacking is a detailed study of stationary solutions in a bounded domain. We plan to report on that in the future.

The paper is organized as follows. In Section 2 we collect the properties of the operators L and A and the properties of the fronts. In Section 3 we prove some Lemmas that allow partial control of Ph and some z -derivative of Ph in terms of $(I - P)h$. In Section 4 we give the energy estimates for the function, the time derivative and the z -derivative, which imply stability and decay of the solution.

2. Spectral Gaps of L and A

In this section we collect all the relevant properties of the operators L and A and also the properties of the fronts.

Lemma 2.1 *There is $\nu_0 > 0$ such that for all $g = (g_1, g_2)$,*

$$\langle g, Lg \rangle_M \leq -\nu_0 \|(I - P)g\|_D^2. \quad (2.1)$$

Proof. Since w_i is bounded from below for $i = 1, 2$, we only need to consider the case when g is a scalar.

Recall (1.2), the null space of L is clearly made of constants (in v) times M . Moreover, L is symmetric with respect to the inner product $(\cdot, \cdot)_M$, so that Lg is orthogonal to the null space of L . We denote by P the projector on the null space of L . Finally, the spectral gap property holds [12]: for any g in the domain of L

$$(g, Lg)_M \leq -\nu((I - P)g, (I - P)g)_M$$

On the other hand, a direct computation yields

$$(g, Lg)_M = - \int_{\mathbb{R}^3} dv M^{-1} |\nabla_v(I - P)g|^2 + 3\beta \int_{\mathbb{R}^3} dv M^{-1} |(I - P)g|^2.$$

We thus conclude our lemma by splitting $(Lg, g)_M = (1 - \epsilon)(Lg, g)_M + \epsilon(Lg, g)_M$ and applying the spectral gap property and the previous identity, for ϵ sufficiently small. \square

By (1.16), it is immediate to check that

$$\hat{\mathcal{F}}(w + \epsilon u) - \hat{\mathcal{F}}(w) = \epsilon^2 \langle Au, u \rangle + o(\epsilon^2).$$

Theorem 2.2. *There exist $\nu > 0$ such that*

$$\langle u, Au \rangle \geq \nu \langle (I - \mathcal{P})u, (I - \mathcal{P})u \rangle,$$

where \mathcal{P} is the projector on $\text{Null } A$:

$$\text{Null } A = \{u \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \mid u = cw', c \in \mathbb{R}\}.$$

Proof. We first characterize $\text{Null } A$. We note that (1.7) imply

$$\frac{u_1^2}{w_1} = - \left(\frac{u_1}{w_1'} \right)^2 \beta w_1' U * w_2', \quad \frac{u_2^2}{w_2} = - \left(\frac{u_2}{w_2'} \right)^2 \beta w_2' U * w_1'.$$

From (1.16), (Au, u) takes the form

$$\begin{aligned} \int_{\mathbb{R}} dz \left[\frac{u_1^2(z)}{w_1(z)} + \frac{u_2^2(z)}{w_2(z)} \right] + 2\beta \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' u_1(z) u_2(z') U(z - z') = \\ -\beta \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \left[\frac{u_1(z)}{w_1'(z)} - \frac{u_2(z')}{w_2'(z')} \right]^2 U(z - z') w_1'(z) w_2'(z'). \quad (2.2) \end{aligned}$$

But, by the monotonicity properties of w_i it follows that $-w_1'(z)w_2'(z')dzdz'$ is a positive measure on $\mathbb{R} \times \mathbb{R}$. Therefore the quadratic form is non negative and vanishes if and only if h is parallel to w' . In particular, this identifies the null space of the operator A .

To establish the spectral gap of A , it is sufficient to prove the lower bound for the normalized operator $\tilde{A}: L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that

$$(\tilde{A}u)_i = \sqrt{w_i}(A(u\sqrt{w}))_i .$$

The explicit form is

$$(\tilde{A}u)_1 = u_1 + \beta\sqrt{w_1}U * (\sqrt{w_2}u_2), \quad (\tilde{A}u)_2 = u_2 + \beta\sqrt{w_2}U * (\sqrt{w_1}u_1) .$$

The corresponding associated quadratic form is

$$\langle u, \tilde{A}u \rangle = \int_{\mathbb{R}} dz(u_1^2 + u_2^2) + 2\beta \int_{\mathbb{R}} dz\sqrt{w_1}u_1U * (u_2\sqrt{w_2}) .$$

The operator \tilde{A} is a bounded symmetric operator on $\mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R})$. From the previous considerations it is also non negative and positive on the orthogonal complement of its null space. The spectral gap for \tilde{A} is established in [6] by adapting an argument already presented in [8]. For completeness, we give a sketch of the proof below.

We decompose the operator as $\tilde{A} = \tilde{A}^0 + K$ where

$$(\tilde{A}^0u)_1 = u_1 + \beta\sqrt{\rho^+\rho^-} U * u_2, \quad (\tilde{A}^0u)_2 = u_2 + \beta\sqrt{\rho^+\rho^-} U * u_1,$$

$$\begin{aligned} (Ku)_1 &= \beta\sqrt{w_1}U * (\sqrt{w_2}u_2) - \beta\sqrt{\rho^+\rho^-} U * u_2 \\ &= \beta \int_{\mathbb{R}} dz' \left[\sqrt{w_1}(z)\sqrt{w_2}(z') - \sqrt{\rho^+\rho^-} \right] U(|z - z'|)u_2(z'), \end{aligned} \quad (2.3)$$

$$\begin{aligned} (Ku)_2 &= \beta\sqrt{w_2}U * (\sqrt{w_1}u_1) - \beta\sqrt{\rho^+\rho^-}U * u_1 \\ &= \beta \int_{\mathbb{R}} dz' \left[\sqrt{w_2}(z)\sqrt{w_1}(z') - \sqrt{\rho^+\rho^-} \right] U(|z - z'|)u_1(z'). \end{aligned} \quad (2.4)$$

The operator \tilde{A}^0 has the spectral gap property. Indeed, consider the equation

$$\tilde{A}^0u = \lambda u + f. \quad (2.5)$$

Denote by $\tilde{u}(\xi)$, $\tilde{f}(\xi)$ and $\tilde{U}(\xi)$ the Fourier transforms of u , f and U . We note that λ is in the resolvent set of \tilde{A}^0 if we can find a unique solution to (2.5), i.e. if the determinant of the matrix

$$\begin{pmatrix} 1 - \lambda & \beta\tilde{U}\sqrt{\rho^+\rho^-} \\ \beta\tilde{U}\sqrt{\rho^+\rho^-} & 1 - \lambda \end{pmatrix}$$

is different from zero for any $\xi \in R$. This happens if λ is such that for all $\xi \in R$

$$(1 - \lambda)^2 - \beta^2(\tilde{U}(\xi))^2\rho^+\rho^- \neq 0.$$

Moreover, by the positivity of U , $|\tilde{U}(\xi)| \leq \tilde{U}(0) = 1$. As a consequence, the spectrum of \tilde{A}^0 is in the interval

$$[1 - \beta\sqrt{\rho^+\rho^-}, 1 + \beta\sqrt{\rho^+\rho^-}].$$

Now, for $\beta > \beta_c$ it is immediate to check that $\beta\sqrt{\rho^+\rho^-} < 1$ and hence the spectrum is contained in $(k, +\infty)$ for some positive k .

We claim that K is compact on \mathcal{H} . Indeed, uniformly for $\|u\| \leq 1$, K satisfies

1. $\forall \epsilon > 0 \quad \exists Z_\epsilon > 0$:

$$\int_{|z| > Z} dz |Ku|^2 < \epsilon, \quad Z > Z_\epsilon$$

2. $\forall \epsilon > 0 \quad \exists \ell_\epsilon > 0$:

$$\int dz |Ku(z + \ell) - Ku(z)|^2 < \epsilon, \quad \ell < \ell_\epsilon.$$

The proof follows trivially from the regularity of the convolution, the fact that U has compact support and the fact that $\lim_{x, y \rightarrow (\pm\infty, \pm\infty)} \sqrt{w_1(x)w_2(y)} = \sqrt{\rho^+\rho^-}$. For the property 2 the boundedness of w'_i and the regularity of U are used. Hence, by Weyl's lemma we have that the spectral gap holds also for \tilde{A} . \square

We are also interested into a lower bound on the norm of $(Au)'$. To this purpose, consider $u = (u_1, u_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with derivative $u' \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Assume u orthogonal to $w' = (w'_1, w'_2)$: $\langle u, w' \rangle = 0$.

We now take the orthogonal decomposition of *each* component of u with respect to the corresponding component of $w' = (w'_1, w'_2)$ in the scalar L^2 inner product. In terms of the vector inner product, by a direct computation, such a process leads to

$$u = \alpha \tilde{w}' + \tilde{u} \tag{2.6}$$

where \tilde{u} is such that

$$\int_{\mathbb{R}} dz \tilde{u}_1 w'_1 = 0 = \int_{\mathbb{R}} dz \tilde{u}_2 w'_2$$

while $\tilde{w}' = (w'_1, -w'_2)$ is orthogonal to w' in the inner product $\langle \cdot, \cdot \rangle$ (note that $w'_2(z) = -w'_1(-z)$) with the coefficient α computed as

$$\alpha = \frac{\langle u, \tilde{w}' \rangle}{N},$$

$N = \langle \tilde{w}', \tilde{w}' \rangle = 2 \int dz (w'_1)^2 = 2 \int dz (w'_2)^2$. We first prove a Lemma for \tilde{u} .

Lemma 2.3 *There is a constant C such that*

$$\|(A\tilde{u})'\|^2 \geq C \|\mathcal{Q}\tilde{u}'\|^2. \tag{2.7}$$

where \mathcal{Q} is the orthogonal projection on the orthogonal complement of w'' .

Proof. We follow the proof in [7]. We have

$$(A\tilde{u})'_i = \frac{d}{dz} \left[\frac{\tilde{u}_i}{w_i} + U * u_j \right] = \left[\frac{\tilde{u}'_i}{w_i} + U * \tilde{u}'_j \right] - \frac{w'_i}{w_i^2} u_i = (A\tilde{u}')_i - \frac{w'_i}{w_i^2} \tilde{u}_i .$$

By integrating over \bar{z} the identity

$$\tilde{u}_i(z) = \tilde{u}_i(\bar{z}) + \int_{\bar{z}}^z ds \tilde{u}'_i(s)$$

multiplied by $w'_i(\bar{z})$, we get

$$\tilde{u}_i(z) = \frac{(-1)^{i+1}}{(\rho^+ - \rho^-)} \int_{-\infty}^{+\infty} d\bar{z} w'_i(\bar{z}) \int_{\bar{z}}^z ds \tilde{u}'_i(s) ,$$

because $\int dz \tilde{u}_i w'_i = 0$.

From above, we can write $(A\tilde{u})'_i$ in terms of an operator $A + K$ acting on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that

$$(A\tilde{u})'_i = (A\tilde{u}')_i + (K\tilde{u}')_i ,$$

$$(Kh)_i(z) := \frac{(-1)^i}{(\rho^+ - \rho^-)} \frac{w'_i}{w_i^2} \int_{-\infty}^{+\infty} dz' w'_i(z') \int_{z'}^z ds h_i(s) .$$

We prove first that

The operator K is compact on L^2 .

Indeed, we show that

– $\forall \varepsilon > 0 \quad \exists Z_\varepsilon > 0$:

$$\int_{|z| > Z} dz |Kh|^2 < \varepsilon, \quad Z > Z_\varepsilon ,$$

– $\forall \varepsilon > 0 \quad \exists \ell_\varepsilon > 0$:

$$\int dz |Kh(z + \ell) - Kh(z)|^2 < \varepsilon, \quad \ell < \ell_\varepsilon .$$

The second is true because of the continuity of the integral. To prove the first, note that

$$\left| \int_{-\infty}^{+\infty} dz' w'_i(z') \int_{z'}^z ds h_i(s) \right| \leq \|h\| \int_{-\infty}^{+\infty} dz |w'_i(z)| \sqrt{|z - z'|} \leq C(1 + |z|) \|h\|$$

(2.8)

so that

$$\int_{|z| > Z} dz \left| \frac{w'_i}{w_i^2}(z) \right|^2 \left| \int_{-\infty}^{+\infty} dz' w'_i(z') \int_{z'}^z ds h_i(s) \right|^2$$

$$\leq C \|h\| \int_{|z| > Z} dz \left| \frac{w'_i}{w_i^2}(z) \right|^2 (1 + |z|)^2 .$$

Then, by the rapid decay property of w'_i ,

$$\int_{|z|>Z} dz |(Kh)_i|^2 \rightarrow 0, \quad Z \rightarrow +\infty,$$

which gives the result. Now,

$$\int_{\mathbb{R}} dz |(A\tilde{u})'|^2 = \int_{\mathbb{R}} dz \tilde{u}' (A^2 + K^*A + AK^* + K^*K) \tilde{u}' .$$

The operator $K^*A + AK^* + K^*K$ is compact because A is bounded and K compact and its null space is spanned by w'' , because by definition of $A + K$

$$0 = (Aw')' = (A + K)w'' .$$

But, A^2 has a strictly positive essential spectrum, hence the result follows from Weyl's lemma. Moreover

$$\int_{\mathbb{R}} dz |(A\tilde{w}')|^2 = \delta > 0,$$

because \tilde{w}' is orthogonal to the null space of A . \square

Theorem 2.4. *For any $u \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, $u' \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that $\langle u, w' \rangle = 0$, there exists a positive constant B such that*

$$\|(Au)'\|^2 \geq B(|\alpha|^2 + \|Q\tilde{u}'\|^2). \quad (2.9)$$

where Q is the projection on the orthogonal complement of w'' . Furthermore, if $u' = Qu'$, then

$$\|(Au)'\|^2 \geq B(|\alpha|^2 + \|\tilde{u}'\|^2). \quad (2.10)$$

Proof. First, we prove that there is a constant C such that, if $u = (1 - \mathcal{P})u$,

$$\|(Au)'\|^2 \geq C(\delta\alpha^2 + \|(A\tilde{u})'\|^2) . \quad (2.11)$$

We introduce the normalized vector ω and its decomposition along w' and the orthogonal complement by setting:

$$\omega = \frac{u}{\delta\alpha^2 + \|(A\tilde{u})'\|^2}; \quad \omega = \eta\tilde{w}' + \tilde{\omega} ,$$

so that equation (2.11) reads as

$$\|(A\omega)'\|^2 \geq C. \quad (2.12)$$

By the decomposition of ω we have

$$\|(A\omega)'\|^2 = \|(A\tilde{\omega})'\|^2 + \delta\eta^2 + 2((A\tilde{\omega})', \eta(A\tilde{w}')') .$$

By definition, ω is such that

$$\|(A\tilde{\omega})'\|^2 + \delta\eta^2 = 1,$$

hence

$$\|(A\omega)'\|^2 = 1 + 2\langle(A\tilde{\omega})', \eta(A\tilde{w}')'\rangle.$$

Suppose now that the inequality (2.12) is not true. Then, for any n we can find $\tilde{\omega}_n$ and η_n such that

$$\|(A[\tilde{\omega}_n + \eta_n \tilde{w}']')\|^2 = 1 + 2\langle(A\tilde{\omega})'_n, \eta_n(A\tilde{w}')'\rangle < \frac{1}{n}.$$

By weak compactness, up to subsequences, there are $\tilde{\omega}_0$ and η_0 such that $\tilde{\omega}_n$ converges weakly to $\tilde{\omega}_0$, $\eta_n \rightarrow \eta_0$. By weak convergence,

$$\langle(A\tilde{\omega}_n)', \eta_n(A\tilde{w}')'\rangle \rightarrow \langle(A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle$$

and

$$\liminf[\|(A\tilde{\omega}_n)'\|^2 + \delta\eta_n^2] + 2\langle(A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle = 0$$

By lower semicontinuity,

$$\|(A\tilde{\omega}_0)'\|^2 + \delta\eta_0^2 \leq \liminf[\|(A\tilde{\omega}_n)'\|^2 + \delta\eta_n^2] = 1$$

Hence,

$$\begin{aligned} 0 &\leq \|(A\omega_0)'\|^2 = \|(A\tilde{\omega}_0)'\|^2 + \delta\eta_0^2 + 2\langle(A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle \\ &\leq 1 + 2\langle(A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle \leq 0. \end{aligned} \quad (2.13)$$

As a consequence,

$$\|(A\omega_0)'\|^2 = 0$$

which implies $\omega_0 = 0$: indeed $\langle\omega_0, w'\rangle = \lim_{n \rightarrow \infty} \langle\omega_n, w'\rangle = 0$ because ω_n is a sequence of vectors orthogonal to w' . Furthermore, since $\omega_n \rightarrow \omega_0 = 0$ weakly, $\eta_n \rightarrow \eta_0 = 0$. Then, $\langle(A\tilde{\omega}_0)', \eta_0(A + \tilde{w}')'\rangle = 0$ in contradiction with last inequality in (2.13). Therefore (2.12) is true and, together with (2.7), implies (2.9).

Finally, to prove (2.10), we notice that if $u' = Qu'$, then by (2.6),

$$\alpha\tilde{w}'' + \tilde{u}' = u' = Qu' = \alpha Q\tilde{w}'' + Q\tilde{u}'.$$

We now show that $Q\tilde{w}'' = \tilde{w}''$, so that the previous identity implies $Q\tilde{u}' = \tilde{u}'$ and hence the result. We have that

$$\langle\tilde{w}'', w''\rangle = (w''_1, w''_1) - (w''_2, w''_2) = 0$$

because $w''_2(z) = w''_1(-z)$. \square

We conclude this Section with a pointwise bound following from previous theorems:

Lemma 2.5 *For any function $u = (u_1, u_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that $\langle u, w' \rangle = 0$ and $\langle u, w'' \rangle = 0$, there is a constant C such that, for any $z \in \mathbb{R}$*

$$|u(z)| \leq C(1 + |z|)\|(Au)'\|. \quad (2.14)$$

Proof. By using the decomposition (2.6), we write, using the notation of Theorem 2.4,

$$u = \alpha \tilde{w}' + \tilde{u}$$

Then the argument leading to (2.8) provides the estimate

$$|\tilde{u}(z)| \leq \int_{-\infty}^{+\infty} d\bar{z} w'(\bar{z}) \int_{\bar{z}}^z dy |\tilde{u}'(y)| \leq (1 + |z|)\|\tilde{u}'\| \leq (1 + |z|)\|(Au)'\|,$$

where the second inequality uses the fast decay of w' and the third one Theorem 2.4. By the same theorem we have also

$$|\alpha| \leq \|(Au)'\|.$$

Since w' decays, we obtain (2.14). \square

3. Estimates of the hydrodynamic part Ph .

We decompose the solution of (1.8) in the component in the null space of L and in the one orthogonal to the null space: $h_i = Ph_i + (I - P)h_i$. We denote by Ma_i the components in the null space of L : $Ph_i = M \int_{\mathbb{R}^3} dv h_i = Ma_i$, so that

$$h_i = a_i M + (I - P)h_i.$$

Since the force $F(h)$ only depends on a , the abuse of notation

$$F_i(a) = -\partial_z U * a_j, \quad i = 1, 2$$

will be used when convenient instead of $F_i(h)$. By using this decomposition in (1.8) we have

$$\begin{aligned} & M [\partial_t a_i + v_z \partial_z a_i - a_i U * w_j' M^{-1} \partial_{v_z} M + \beta v_z w_i U * \partial_z a_j] \\ &= -\partial_t (I - P)h_i - G_i (I - P)h_i - F_i(a) \partial_{v_z} h_i + L(I - P)h_i, \end{aligned} \quad (3.1)$$

with G_i defined in (1.9), which we rewrite for reader's convenience:

$$G_i h_i = v_z \partial_z h_i - (U * w_j') \partial_{v_z} h_i + \left(U * \partial_z \int_{\mathbb{R}^3} dv h_j(\cdot, v, t) \right) \beta v_z M w_i \quad (3.2)$$

We define

$$\mu_i = \frac{a_i}{w_i} + \beta U * a_j := (Aa)_i$$

so that

$$\partial_z \mu_i = \frac{1}{w_i} \partial_z a_i - a_i \frac{w'_i}{w_i^2} + \beta U * \partial_z a_j$$

By using the equation for the front (1.7) we can write the equation (3.1) as

$$\begin{aligned} & M[\partial_t a_i + v_z w_i \partial_z \mu_i] \\ &= -\partial_t(I-P)h_i - G_i(I-P)h_i - F_i(a)\partial_{v_z} h_i + L(I-P)h_i. \end{aligned} \quad (3.3)$$

By integrating (3.3) over the velocity, since $\int_{\mathbb{R}^3} dv \partial_t(I-P)h_i = 0$, we have

$$\partial_t a_i = - \int_{\mathbb{R}^3} dv G_i(I-P)h_i$$

and, by Definition (3.2) of G_i ,

$$\partial_t a_i = -\partial_z \int_{\mathbb{R}^3} dv v_z (I-P)h_i \quad (3.4)$$

By integrating (3.3) over the velocity after multiplication by v_z we obtain

$$\begin{aligned} T w_i \partial_z \mu_i &= - \int_{\mathbb{R}^3} dv v_z \partial_t(I-P)h_i - \int_{\mathbb{R}^3} dv v_z G_i(I-P)h_i \\ &\quad - \int_{\mathbb{R}^3} dv v_z F_i(a)\partial_{v_z} h_i + \int_{\mathbb{R}^3} dv v_z L(I-P)h_i. \end{aligned}$$

Moreover, by integrating by parts,

$$\begin{aligned} \int_{\mathbb{R}^3} dv v_z G_i(I-P)h_i &= \int_{\mathbb{R}^3} dv v_z^2 \partial_z(I-P)h_i + U * w'_j \int_{\mathbb{R}^3} dv (I-P)h_i \\ &= \partial_z \int_{\mathbb{R}^3} dv v_z^2 (I-P)h_i. \end{aligned}$$

Hence

$$\begin{aligned} T w_i \partial_z \mu_i &= - \int_{\mathbb{R}^3} dv v_z \partial_t(I-P)h_i - \partial_z \int_{\mathbb{R}^3} dv v_z^2 (I-P)h_i \\ &\quad + \int_{\mathbb{R}^3} dv v_z L(I-P)h_i + F_i(a)a_i. \end{aligned} \quad (3.5)$$

We define

$$\ell_i^a = \int_{\mathbb{R}^3} dv v_z (I-P)h_i, \quad \ell_i^b = \int_{\mathbb{R}^3} dv v_z^2 (I-P)h_i, \quad m_i = \int_{\mathbb{R}^3} dv v_z L(I-P)h_i.$$

By integrating twice by parts we get the identity:

$$m_i = - \int_{\mathbb{R}^3} dv M \partial_{v_z} \left(\frac{(I-P)h_i}{M} \right) = \beta \int_{\mathbb{R}^3} dv v_z (I-P)h_i = \beta \ell_i^a.$$

The following estimates are a simple consequence of (3.4) and (3.5).

$$\|\partial_t a_i\| = \|\partial_z \ell_i^a\|$$

$$\|\partial_z \mu_i\| \leq \|\partial_t \ell_i^a\| + \|\partial_z \ell_i^b\| + \|m_i\| + \|\partial_z a_i\| \|a_i\|$$

From the definition, we have

$$|\ell_i^a| = \left| \int_{\mathbb{R}^3} dv v_z \sqrt{M} \frac{1}{\sqrt{M}} (I - P) h_i \right| \leq C \left[\int dv \frac{|(I - P) h_i|^2}{M} \right]^{\frac{1}{2}}.$$

This and the fact that w is bounded from above and below give

$$\|\ell_i^a\|^2 \leq \rho^+ \int_{\mathbb{R}} dz \frac{1}{w_i} |\ell_i^a|^2.$$

Hence,

$$\sum_{i=1}^2 \|\ell_i^a\| \leq C \|(I - P)h\|_M.$$

Motivated by Theorem 3.2 below, we introduce the following decomposition:

$$\begin{aligned} \partial_z \mu_i^{(1)} &= -\frac{1}{T w_i} \left[\int_{\mathbb{R}^3} dv v_z (I - P) \partial_t h_i - \int_{\mathbb{R}^3} dv v_z \right. \\ &\quad \left. + L(I - P) h_i - F_i(a) a_i \right] + \partial_z \left(\frac{1}{T w_i} \right) \int_{\mathbb{R}^3} dv v_z^2 (I - P) h_i, \end{aligned} \quad (3.6)$$

$$\partial_z \mu_i^{(2)} = -\partial_z \left(\frac{1}{T w_i} \int_{\mathbb{R}^3} dv v_z^2 (I - P) h_i \right) \quad (3.7)$$

so that $\mu_i^{(1)} + \mu_i^{(2)} = \mu_i$. We define $a^{(k)}$, $k = 1, 2$, by setting $\mu_i^{(1)} = (Aa^{(1)})_i$, $\mu_i^{(2)} = (Aa^{(2)})_i$. Since the null space of A is given by $\alpha w' = (\alpha w'_1, \alpha w'_2)$ for $\alpha \in \mathbb{R}$, the equation

$$\mu_i = A g_i$$

has solutions if and only if

$$\sum_{i=1}^2 \int_{\mathbb{R}} dz w'_i \mu_i = 0$$

and they are of the form

$$g_i = (A^{-1} \mu)_i + \theta w'_i$$

where $(A^{-1} \mu)$ is the unique solution orthogonal to the null space of A and $\theta \in \mathbb{R}$. Therefore, we need to show that $\mu^{(i)}$ are orthogonal to the null space of A . We shall prove this at the end of the proof of Lemma 3.1. Moreover, we can always choose $\theta = 0$ since $a = a^{(1)} + a^{(2)}$ and a does not have component on the null space of A . In fact, a has by assumption at time zero the same symmetry property as w and it is preserved in time. This implies that at any time a is orthogonal to w' and hence has no component in the

null space of A . This is one of the crucial points where we use the symmetry assumption on the initial perturbation.

We now estimate the L^2 norm $\|\partial_z a^{(1)}\|$. To this end, we first prove that $\mathcal{Q}\partial_z a_i^{(1)} = \partial_z a_i^{(1)}$ which is equivalent to show that $\sum_{i=1}^2 \int \partial_z a_i^{(1)} w_i'' = 0$.

Lemma 3.1 *If $h_1(z, v, t) = h_2(-z, Rv, t)$, then $\langle \partial_z a^{(k)}, w'' \rangle = 0$, $k = 1, 2$.*

Proof: We notice that this property is true for $\partial_z a$ because of the symmetry properties of the solution. In fact, $\partial_z a_1(z) = -\partial_z a_2(-z)$ and $w_1''(z) = w_2''(-z)$. We are left with proving that the same symmetry property holds for each $a^{(j)}$. It is then enough to prove that for $a^{(2)}$. We have that

$$a^{(2)} = -A^{-1} \left[\frac{1}{Tw_i} \int_{\mathbb{R}^3} dv v_z^2 (I - P) h \right]$$

and since A does not change the symmetry properties it is enough to prove that

$$\left(\frac{1}{w_1} \int_{\mathbb{R}^3} dv v_z^2 (I - P) h_1 \right) (z) = \left(\frac{1}{w_2} \int_{\mathbb{R}^3} dv v_z^2 (I - P) h_2 \right) (-z).$$

By using the properties of h we have that the left hand side is equal to

$$\frac{1}{w_1(z)} \int_{\mathbb{R}_+^3} dv v_z^2 (I - P) [h_1(v, z) + h_2(v, -z)],$$

with \mathbb{R}_+^3 the set of velocities with $v_z \geq 0$, and the right hand side to

$$\frac{1}{w_2(-z)} \int_{\mathbb{R}_+^3} dv v_z^2 (I - P) [h_1(v, z) + h_2(v, -z)].$$

The symmetry properties of w_i imply the result. The same argument also shows that $\mu^{(2)}$ is orthogonal to the kernel of A and hence $\mu^{(1)}$ has the same property. This follows from the fact that μ is in the range of A which is orthogonal to the null space of A because it is a symmetric operator on L^2 . \square

Theorem 3.2. *We have*

$$\sum_{i=1}^2 \|a_i^{(2)}\| \leq C \|(I - P)h\|_M \leq C \|h\|_D$$

Moreover, there is $\delta_0 > 0$ such that, if $\|h\|_M \leq \delta_0$,

$$\sum_{i=1}^2 \|\partial_z a_i^{(1)}\| \leq C [\|(I - P)h\|_M + \|(I - P)\partial_t h\|_M]$$

Remark: Note that the estimate of $\partial_z a_i^{(1)}$ do not involve $\partial_z(I-P)h$. This is the main reason for the decomposition (3.6), (3.7).

Proof: From (3.7), by integration over z , since $\mu_i \rightarrow 0$ as $z \rightarrow \pm\infty$,

$$\mu_i^{(2)} = -\frac{1}{Tw_i} \int_{\mathbb{R}^3} dv v_z^2 (I-P)h_i = \frac{1}{Tw_i} \ell_i^b$$

which implies

$$\sum_{i=1}^2 \|\mu_i^{(2)}\| \leq C\|(I-P)h\|_M$$

Moreover, $\mu_i^{(2)} = (Aa^{(2)})_i$ so that we have also, by Theorem 2.2 and the fact that $(I-P)a^{(k)} = a^{(k)}$, $k = 1, 2$,

$$\sum_{i=1}^2 \|a_i^{(2)}\| \leq C\|(I-P)h\|_M .$$

From (3.6) we get

$$\sum_{i=1}^2 \|\partial_z \mu_i^{(1)}\| \leq C\|(I-P)\partial_t h\|_M + C\|(I-P)h\|_M + \sup_i \|F_i(a)\|_{L^\infty} \sum_{i=1}^2 \|a_i\| .$$

Now, by the regularity properties of U ,

$$\|F_i(a)\|_{L^\infty} = \|U * \partial_z a_j^{(1)} + \partial_z U * a_j^{(2)}\|_{L^\infty} \leq C\|\partial_z a_j^{(1)}\| + C\|a_j^{(2)}\| .$$

To apply Theorem 2.4 we need to show that a is orthogonal to w' . We notice that the front w is symmetric under the exchange $1 \rightarrow 2$ while the derivatives w'_i are antisymmetric. On the other hand, as already observed, a has at time zero the same symmetry properties as w and this implies that the component of a on the null space of A is zero at any time. In addition, by Lemma 3.1, we can apply Theorem 2.4 to get

$$\begin{aligned} \|\partial_z a^{(1)}\| &\leq C \left[\|(I-P)\partial_t h\|_M + \|(I-P)h\|_M + \|Ph\|_M (\|\partial_z a^{(1)}\| + \|a^{(2)}\|) \right] \\ &\leq C \left[\|(I-P)\partial_t h\|_M + \|(I-P)h\|_M + \|h\|_M (\|\partial_z a^{(1)}\| + \|(I-P)h\|_M) \right] . \end{aligned}$$

Then, for δ_0 small enough

$$\|\partial_z a^{(1)}\| \leq C \left[\|(I-P)\partial_t h\|_M + C\|(I-P)h\|_M + \delta_0\|(I-P)h\|_M \right]$$

which proves Theorem 3.2. \square

As a consequence we have also

$$\sum_{i=1}^2 \|F(h_i)\|_{L^\infty} \leq C \left[\|(I-P)\partial_t h\|_M + \|(I-P)h\|_M \right] . \quad (3.8)$$

From now on we use the more explicit notation $Ma_h = Ph$. Moreover we use the previous decomposition: $a_h = a_h^{(1)} + a_h^{(2)}$. Furthermore $\|b\|_\gamma$ will denote the γ -weighted L^2 -norm $\|z_\gamma b\|$.

Lemma 3.3 *Let $0 \leq \gamma \leq 1$. Then*

$$\|a_h^{(2)}\|_\gamma \leq C \|h\|_{D,\gamma}, \quad (3.9)$$

Moreover, there is $\delta_0 > 0$ such that, if $\|h\|_M \leq \delta_0$, then for any $z \in \mathbb{R}$,

$$|a_h^{(1)}(z)| \leq (1 + |z|) (\|h\|_D + \|\partial_t h\|_D) \quad (3.10)$$

and

$$\|\partial_z a_h^{(1)}\|_\gamma \leq C (\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma}). \quad (3.11)$$

Furthermore, if we have also $\gamma \leq \frac{1}{8}$,

$$\int dz \frac{z^2}{(1+z^2)^{2-2\gamma}} |a_h|^2 \leq C (\|h\|_{D,\gamma}^2 + \|\partial_t h\|_{D,\gamma}^2). \quad (3.12)$$

Proof: We introduce the *commutator* $[z_\gamma, A]$ defined as follows: for any function $a = (a_1, a_2)$

$$[z_\gamma, A]a = z_\gamma Aa - A(z_\gamma a),$$

which, by the definition of A , reduces to

$$[z_\gamma, U]a \equiv z_\gamma U * a - U * (z_\gamma a)$$

The commutator can be estimated by taking into account the property of the convolution and the fact that U is of finite range. Indeed, it is easy to check that, if $|z - z'| < R$, then, with the notation $z'_\gamma = (1 + z'^2)^\gamma$, we have

$$|z_\gamma - z'_\gamma| \leq \gamma(z_{\gamma-\frac{1}{2}} + z'_{\gamma-\frac{1}{2}})|z - z'| \leq C\gamma z'_{\gamma-\frac{1}{2}}|z - z'| \quad (3.13)$$

for some constant C . Therefore

$$\begin{aligned} |[z_\gamma, U]a(z)| &= \left| \int dz' U(|z - z'|)(z_\gamma - z'_\gamma)a(z') \right| \\ &\leq C \int dz' U(|z - z'|)|z - z'| z'_{\gamma-\frac{1}{2}} a(z') \end{aligned}$$

Hence

$$\|[z_\gamma, U]a\| \leq C \|z_{\gamma-\frac{1}{2}} a\| \quad (3.14)$$

and in particular,

$$\|[z_\gamma, A]a_h^{(2)}\| \leq C \|z_{\gamma-\frac{1}{2}} a_h^{(2)}\|.$$

Moreover

$$z_\gamma A a_h^{(2)} = -z_\gamma \frac{1}{T w_i} \int dv v_z^2 (I - P) h_i.$$

which implies

$$\|z_\gamma A a_h^{(2)}\| \leq C \|h\|_{D,\gamma}.$$

The last two estimates together imply

$$\|A(z_\gamma a_h^{(2)})\| \leq C\|h\|_{D,\gamma} + \|a_h^{(2)}\|_{\gamma-\frac{1}{2}}.$$

Since $z_\gamma a_h^{(2)}$ has the same symmetry properties of $a_h^{(2)}$, it is orthogonal to w' as well and we can use Theorem 2.2 to deduce

$$\|a_h^{(2)}\|_\gamma \leq C(\|h\|_{D,\gamma} + \|a_h^{(2)}\|_{\gamma-\frac{1}{2}}).$$

By repeating the argument with $\gamma - \frac{1}{2}$ instead of γ and using the fact that $z_{\gamma-1} < 1$ for $\gamma \leq 1$, by the first part of Theorem 3.2, we obtain (3.9).

To prove the second statement it is enough to note that $a_h^{(1)}$ is orthogonal to w' by construction and to w'' by symmetry (Lemma 3.1). Hence we have, by Lemma 2.5,

$$|a_h^{(1)}(z)| \leq C(1 + |z|)\|(Aa_h^{(1)})'\|, \quad (3.15)$$

and, by Theorem 3.2, we obtain (3.10).

To estimate $\partial_z a_h^{(1)}$ we note that $\partial_z (Aa_h^{(1)})_i = (A\partial_z a_h^{(1)})_i - \frac{w'_i}{w_i^2}(a_h^{(1)})_i$.

Therefore

$$z_\gamma (A\partial_z a_h^{(1)})_i = z_\gamma \partial_z (Aa_h^{(1)})_i + \frac{z_\gamma w'_i}{w_i^2} (a_h^{(1)})_i. \quad (3.16)$$

Clearly, since w'_i decays exponentially, by (3.10), we have the following estimate for the second term in (3.16)

$$\left\| \frac{z_\gamma w'_i}{w_i^2} (a_h^{(1)})_i \right\| \leq C(\|h\|_D + \|\partial_t h\|_D)$$

We examine now the first term in (3.16). We deduce from equation (3.6)

$$\|z_\gamma \partial_z (Aa_h^{(1)})\| \leq C(\|h\|_{D,\gamma} + C\|\partial_t h\|_{D,\gamma} + \|z_\gamma F(a_h)\|_{L^\infty} \|a_h\|).$$

We further split $\|z_\gamma F(a_h)\|_{L^\infty}$ in the last term as

$$\begin{aligned} & \|z_\gamma F(a_h^{(2)})\|_{L^\infty} + \|z_\gamma U * (\partial_z a_h^{(1)})\|_{L^\infty} \\ & \leq \|F(z_\gamma(a_h^{(2)}))\|_{L^\infty} + \|[z_\gamma, \partial_z U]a_h^{(2)}\|_{L^\infty} \\ & + \|U * (z_\gamma \partial_z a_h^{(1)})\|_{L^\infty} + \|[z_\gamma, U]\partial_z a_h^{(1)}\|_{L^\infty} \\ & \leq \left(\|z_\gamma a_h^{(2)}\| + \|z_\gamma \partial_z a_h^{(1)}\| \right) \leq C \left(\|h\|_{D,\gamma} + \|z_\gamma \partial_z a_h^{(1)}\| \right). \end{aligned}$$

We have used (3.9) in the last inequality. The commutators are estimated as before in (3.14), leading to a term

$$\|z_{\gamma-\frac{1}{2}} \partial_z a_h^{(1)}\| + \|z_{\gamma-\frac{1}{2}} a_h^{(2)}\| \leq \|z_\gamma \partial_z a_h^{(1)}\| + \|h\|_{D,\gamma},$$

where we have used again (3.9). We hence conclude that, for $\|a_h\|$ small,

$$\begin{aligned} \|A(z_\gamma \partial_z a_h^{(1)})\| &\leq \|z_\gamma A \partial_z a_h^{(1)}\| + \|[z_\gamma, A] \partial_z a_h^{(1)}\| \\ &\leq C \left(\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma} + \|z_\gamma \partial_z a_h^{(1)}\| \|a_h\| \right). \end{aligned}$$

We now decompose along the null space of A and its orthogonal complement in order to use the spectral gap of A : Denote by

$$\tau = w' \|w'\|^{-1}$$

the unit vector in the direction w' .

By the decay of w' , for any L^2 function q , we get

$$|\langle z_\gamma q, \tau \rangle| \leq \|q\|. \quad (3.17)$$

Hence we have

$$\begin{aligned} \|z_\gamma \partial_z a_h^{(1)}\| &\leq \|z_\gamma \partial_z a_h^{(1)} - \langle z_\gamma \partial_z a_h^{(1)}, \tau \rangle \tau\| + \|\langle z_\gamma \partial_z a_h^{(1)}, \tau \rangle \tau\| \\ &\leq C \|A(z_\gamma \partial_z a_h^{(1)})\| + C \{\|h\|_D + \|\partial_t h\|_D\} \\ &\leq C \{\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma}\} + C \|z_\gamma \partial_z a_h^{(1)}\| \|a_h\| \end{aligned}$$

Putting estimates together, for $\|a_h\|$ small,

$$\|z_\gamma \partial_z (A a_h^{(1)})\| \leq C \{\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma}\} \quad (3.18)$$

and hence we deduce (3.11).

To prove (3.12), first note that the contribution due to $a_h^{(2)}$ is easily bounded by (3.9). As for the contribution due to $a_h^{(1)}$, since we can use the pointwise estimate (3.10), the key is to estimate $\frac{z a_h^{(1)}}{(1+z^2)^{1-\gamma}}$ for z large.

In fact, let $\chi(z)$ be a smooth cutoff function with $\chi(z) \equiv 1$ for $|z| \geq k$, for k large and $\chi(z) \equiv 0$ for $|z| \leq k-1$. We have for the contribution from $a_h^{(1)}$ due to $|z| \leq k$, by (3.10)

$$\begin{aligned} &\int dz \frac{z^2}{(1+z^2)^{2-2\gamma}} \{1-\chi\} |a_h^{(1)}|^2 \\ &\leq \left(\|h\|_D^2 + \|\partial_t h\|_D^2 \right) \int_{|z| \leq k} dz \frac{z^2(1+|z|)^2}{(1+z^2)^{2-2\gamma}} \leq C_k \left(\|h\|_D^2 + \|\partial_t h\|_D^2 \right). \end{aligned}$$

We now consider the contribution for $|z|$ large. We have:

$$\begin{aligned} &\int dz \frac{\chi z^2 |a_h^{(1)}|^2}{(1+z^2)^{2-2\gamma}} = - \int dz \frac{d}{dz} \left(\frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} \right) z \chi |a_h^{(1)}|^2 \\ &= \int dz \frac{1}{2(1-\gamma)(1+z^2)^{1-2\gamma}} z \chi' |a_h^{(1)}|^2 + \int dz \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} \chi |a_h^{(1)}|^2 \\ &+ \int dz \frac{1}{(1-2\gamma)(1+z^2)^{1-2\gamma}} \chi a_h^{(1)} \partial_z a_h^{(1)}. \end{aligned}$$

Therefore,

$$\int dz \left(\frac{z^2}{(1+z^2)^{2-2\gamma}} - \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} \right) \chi |a_h^{(1)}|^2 = \int dz \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} z \chi' |a_h^{(1)}|^2 + \int dz \frac{1}{(1-2\gamma)(1+z^2)^{1-2\gamma}} \chi a_h^{(1)} \partial_z a_h^{(1)}.$$

For $\gamma \leq \frac{1}{8}$ and $|z| > k$,

$$\frac{z^2}{(1+z^2)^{2-2\gamma}} - \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} \geq \frac{z^2}{4(1+z^2)^{2-2\gamma}}.$$

Since $\chi' \equiv 0$ for $|z| \geq k$, by using again (3.10) to bound the term with χ' , we obtain:

$$\int dz \frac{z^2}{4(1+z^2)^{2-2\gamma}} \chi |a_h^{(1)}|^2 \leq \frac{1}{8} \int dz \frac{z^2}{(1+z^2)^{2-2\gamma}} \chi |a_h^{(1)}|^2 + C \|z_\gamma \partial_z a_h^{(1)}\|^2 + C_k \left(\|h\|_D^2 + \|\partial_t h\|_D^2 \right).$$

We thus deduce (3.12) by using (3.11) and conclude the proof of Lemma 3.3. \square

It will be important in the energy estimate in next section, and in particular in the proof of Lemma 4.4, to bound $\partial_z(z_\gamma \partial_z a^{(1)})$ in terms of at most one space derivative of h . To this end it is convenient to introduce the quantity $a_h^{(3)}$ defined by the positions:

$$(Aa_h^{(3)})_i \equiv -\frac{1}{T w_i} \int dv v_z \partial_t (I - P) h_i, \quad \langle a_h^{(3)}, w' \rangle = 0. \quad (3.19)$$

We note that, by Theorem 2.2 and the orthogonality condition,

$$\|a_h^{(3)}\| \leq C \|\partial_t h\|_D. \quad (3.20)$$

We have:

Lemma 3.4 *There is $\delta_0 > 0$ such that, if $\|a_h\| + \|\partial_z a_h\| \leq \delta_0$, the following estimates*

$$\|\partial_z a_h^{(2)}\|_\gamma \leq C (\|\partial_z h\|_{D,\gamma} + \|h\|_D), \quad (3.21)$$

$$\|a_h^{(3)}\|_\gamma \leq C \|\partial_t h\|_{D,\gamma} \quad (3.22)$$

hold for $0 \leq \gamma \leq 1$. Moreover, if $\|\partial_z a_h^{(1)}\|_{\frac{1}{2}} \leq \eta$ for some finite constant η , then there is C_η such that

$$\|\partial_z [z_\gamma \partial_z a_h^{(1)} - z_\gamma a_h^{(3)}]\| \leq C_\eta \left(\|\partial h\|_{D,\gamma} + \|h\|_{D,\gamma-\frac{1}{2}} \right). \quad (3.23)$$

Proof. For notational simplicity, we denote

$$a_\gamma = z_\gamma \partial_z a_h$$

and similar meaning will have $a_\gamma^{(k)}$, $k = 1, \dots, 3$. We need to estimate $\|a_\gamma^{(3)}\|$, $\|a_\gamma^{(2)}\|$ and $\|\partial_z a_\gamma^{(1)}\|$.

First of all, we prove (3.22). From the definition of $Aa_h^{(3)}$ we have

$$\begin{aligned} \|A(z_\gamma a_h^{(3)})\| &\leq \|z_\gamma Aa_h^{(3)}\| + \|[A, z_\gamma]a_h^{(3)}\| \\ &\leq \|\partial_t h\|_{D,\gamma} + C\|z_{\gamma-\frac{1}{2}}a_h^{(3)}\| \\ &\leq \|\partial_t h\|_{D,\gamma} + C\|a_h^{(3)}\|_{\gamma-\frac{1}{2}}. \end{aligned}$$

We then decompose $z_\gamma a_h^{(3)}$ along the direction τ (recall that $\tau = w'\|w'\|^{-1}$) and its orthogonal complement $(z_\gamma a_h^{(3)})^\perp$:

$$z_\gamma a_h^{(3)} = \langle z_\gamma a_h^{(3)}, \tau \rangle \tau + (z_\gamma a_h^{(3)})^\perp.$$

We deduce, again by Theorem 2.2,

$$\|(z_\gamma a_h^{(3)})^\perp\| \leq C\|\partial_t h\|_{D,\gamma} + C\|a_h^{(3)}\|_{\gamma-\frac{1}{2}}.$$

The component of $z_\gamma a_h^{(3)}$ along τ can be bounded by using (3.17) and (3.20). The proof of (3.22) is completed by repeating the argument with γ replaced by $\gamma - \frac{1}{2}$ and applying the bound to $\|a_h^{(3)}\|_{\gamma-\frac{1}{2}}$.

We now turn to (3.21). Note that

$$\begin{aligned} Aa_\gamma^{(2)} &= A(z_\gamma \partial_z a_h^{(2)}) \\ &= z_\gamma A(\partial_z a_h^{(2)}) - [z_\gamma, A]\partial_z a_h^{(2)} \\ &= z_\gamma \partial_z Aa_h^{(2)} + z_\gamma \frac{w'}{w^2} a_h^{(2)} - [z_\gamma, U]\partial_z a_h^{(2)}. \end{aligned} \tag{3.24}$$

Clearly, by the decay of w' , $\|z_\gamma \frac{w'}{w^2} a_h^{(2)}\| \leq C\|a_h^{(2)}\| \leq C\|h\|_D$ by Theorem 3.2. Since

$$\begin{aligned} z_\gamma \partial_z (Aa_h^{(2)})_i &= -z_\gamma \partial_z \left(\frac{1}{T w_i} \int dv v_z^2 (I - P) h_i \right) = \\ &= -z_\gamma \left(\frac{1}{T w_i} \int dv v_z^2 (I - P) \partial_z h_i \right) + z_\gamma \frac{w'_i}{T (w_i)^2} \int dv v_z^2 (I - P) h_i, \end{aligned} \tag{3.25}$$

again by Theorem 3.2 and the decay of w' it follows that

$$\|z_\gamma \partial_z Aa_h^{(2)}\| \leq C \left(\|\partial_z h\|_{D,\gamma} + \|h\|_D \right). \tag{3.26}$$

By (3.13) we have for the commutator $[z_\gamma, U]\partial_z a_h^{(2)}$:

$$\left\| \int dz' U(|z - z'|)(z_\gamma - z'_\gamma) \partial_z a_h^{(2)}(z') \right\| \leq C \|z_{\gamma - \frac{1}{2}} \partial_z a_h^{(2)}\|. \quad (3.27)$$

We therefore can decompose

$$a_\gamma^{(2)} = \langle a_\gamma^{(2)}, \tau \rangle \tau + (a_\gamma^{(2)})^\perp.$$

By (3.17) and Theorem 3.2 we have

$$\|\langle a_\gamma^{(2)}, \tau \rangle \tau\| \leq \|a_h^{(2)}\| \leq C \|h\|_D.$$

But $\|(a_\gamma^{(2)})^\perp\|$ is bounded by using the spectral gap of A and the inequality

$$\|Aa_\gamma^{(2)}\| \leq C \left(\|h\|_D + \|\partial_z h\|_{D,\gamma} + \|z_{\gamma - \frac{1}{2}} \partial_z a_h^{(2)}\| \right), \quad (3.28)$$

Collecting terms and iterating once the inequality, as before, we deduce (3.21).

Finally, to estimate $\partial_z(a_\gamma^{(1)} - z_\gamma a_h^{(3)})$, we use the commutation relation

$$A\partial_z a_i = \partial_z(Aa)_i + \frac{w'_i}{w_i^2} a_i \quad (3.29)$$

to get

$$\begin{aligned} (Aa_\gamma^{(1)})_i &= \left(A(z_\gamma \partial_z a_h^{(1)}) \right)_i \\ &= z_\gamma \partial_z (Aa_h^{(1)})_i - ([z_\gamma, A] \partial_z a_h^{(1)})_i + \frac{z_\gamma w'_i (a_h^{(1)})_i}{w_i^2} \end{aligned}$$

By equation (3.6) and the definition of $Aa_h^{(3)}$

$$\begin{aligned} z_\gamma \partial_z (Aa_h^{(1)})_i &= \\ z_\gamma (Aa_h^{(3)})_i + z_\gamma \left[\frac{1}{Tw_i} \left[\int_{\mathbb{R}^3} dv v_z L(I - P) h_i + F_i(h)(a_h)_i \right] \right. \\ &\quad \left. + \partial_z \left(\frac{1}{Tw_i} \right) \int_{\mathbb{R}^3} dv v_z^2 (I - P) h_i \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} (Aa_\gamma^{(1)})_i - z_\gamma (Aa_h^{(3)})_i &= \left[\frac{z_\gamma}{Tw_i} \left[\int_{\mathbb{R}^3} dv v_z L(I - P) h_i + F(h_j)(a_h)_i \right] \right. \\ &\quad \left. + \partial_z \left(\frac{1}{Tw_i} \right) \int_{\mathbb{R}^3} dv v_z^2 (I - P) h_i \right] - ([z_\gamma, A] \partial_z a_h^{(1)})_i + \frac{z_\gamma w'_i (a_h^{(1)})_i}{w_i^2}. \end{aligned}$$

Using again the commutation relation (3.29) and the previous relation, we find

$$\begin{aligned}
& (A\partial_z(a_\gamma^{(1)} - z_\gamma a_h^{(3)}))_i \\
&= \partial_z(Aa_\gamma^{(1)} - (A(z_\gamma a_h^{(3)}))_i) - \frac{w'_i}{w_i^2}(a_\gamma^{(1)} - z_\gamma a_h^{(3)})_i \\
&= \partial_z \left[(Aa_\gamma^{(1)})_i - z_\gamma (Aa_h^{(3)})_i - ([z_\gamma, A]a_h^{(3)})_i \right] - \frac{w'_i}{w_i^2}(a_\gamma^{(1)} - z_\gamma a_h^{(3)})_i \\
&= \partial_z \left\{ z_\gamma \left[\frac{1}{Tw_i} \left(\int_{\mathbb{R}^3} dv v_z L(I - P)h_i + F_i(a_h)(a_h)_i \right) \right. \right. \\
&\quad \left. \left. + \partial_z \left(\frac{1}{Tw_i} \right) \int_{\mathbb{R}^3} dv v_z^2 (I - P)h_i \right] - ([z_\gamma, A]\partial_z a_h^{(1)})_i - \frac{z_\gamma w'_i (a_h^{(1)})_i}{w_i^2} \right. \\
&\quad \left. - ([z_\gamma, A]\partial_z a_h^{(1)})_i - ([A, z_\gamma]a_h^{(3)})_i \right\} + \frac{w'_i}{w_i^2}(a_\gamma^{(1)} - z_\gamma a_h^{(3)})_i
\end{aligned}$$

The terms involving the commutator can be estimated by moving the z -derivative on the potential U inside the convolution.

We only need to estimate $\partial_z \left(\frac{z_\gamma}{Tw_i} F_i(a_h)(a_h)_i \right)$, all the other terms being estimated by arguments already used. We expand it as

$$\partial_z \left(\frac{z_\gamma}{Tw_i} \right) F_i(a_h)(a_h)_i - \frac{z_\gamma}{Tw_i} [(\partial_z U * \partial_z(a_h)_j)(a_h)_i - (U * \partial_z(a_h)_j)\partial_z(a_h)_i]$$

The first term is bounded by

$$\left(\|z_{\gamma-\frac{1}{2}} a_h^{(2)}\| + \|z_{\gamma-\frac{1}{2}} \partial_z a_h^{(1)}\| \right) \|a_h\|$$

We modify the last two terms above (up to the factor $(Tw_i)^{-1}$) as follows:

$$\begin{aligned}
& - (\partial_z U * z_\gamma \partial_z(a_h)_j)(a_h)_i - (U * z_\gamma \partial_z(a_h^{(2)})_j)\partial_z(a_h)_i \\
& - (U * z_{\gamma-\frac{1}{2}}(\partial_z a_h^{(1)})_j)z_{\frac{1}{2}}\partial_z(a_h)_i + [\partial_z U, z_\gamma]\partial_z(a_h)_j(a_h)_i \\
& + ([U, z_\gamma]\partial_z(a_h^{(2)})_j + [U, z_{\gamma-\frac{1}{2}}]\partial_z(a_h^{(1)})_j)\partial_z(a_h)_i
\end{aligned}$$

The L^2 norms of the last two terms are bounded by

$$\begin{aligned}
& \left(\|z_{\gamma-\frac{1}{2}} a_h^{(2)}\| + \|z_{\gamma-\frac{1}{2}} \partial_z a_h^{(1)}\| \right) \left(\|a_h\| + \|\partial_z a_h\| \right) \\
& \leq \delta_0 \left(\|h_{D, \gamma-\frac{1}{2}}\| + \|\partial_t h_{D, \gamma-\frac{1}{2}}\| \right)
\end{aligned}$$

The inequality follows from Lemma 3.3 and the smallness assumption $\|a_h\| + \|\partial_z a_h\| \leq \delta_0$. The contribution from $a_h^{(2)}$ to the first term is easily bounded

by the first part of Lemma 3.4. We write the contribution to the first term due to $a_h^{(1)}$ (up to the minus sign) as

$$\left(U * \partial_z \left((a_\gamma^{(1)})_j - z_\gamma (a_h^{(3)})_j \right) + \partial_z U * z_\gamma (a_h^{(3)})_j \right) (a_h)_i$$

Finally, we get

$$\begin{aligned} \left\| \partial_z \left(\frac{z_\gamma}{T w_i} F_i(a_h)(a_h)_i \right) \right\| &\leq \delta_0 (\|h_{D, \gamma - \frac{1}{2}}\| + \|\partial_t h\|_{D, \gamma - \frac{1}{2}}) \\ &+ \left(\|\partial_z (a_\gamma^{(1)} - z_\gamma a_h^{(3)})\| + \|z_\gamma a_h^{(3)}\| + \|z_\gamma \partial_z a_h^{(2)}\| \right) \\ &\times (\|a_h\| + \|\partial_z a_h\|) + \|\partial_z a^{(1)}\|_{\gamma - \frac{1}{2}} \|\partial_z a_h\|_{\frac{1}{2}}. \end{aligned}$$

We use (3.21), (3.22) and Lemma 3.3 to get

$$\|z_\gamma a_h^{(3)}\| + \|z_\gamma \partial_z a_h^{(2)}\| + \|\partial_z a^{(1)}\|_{\gamma - \frac{1}{2}} \leq \|h\|_{D, \gamma - \frac{1}{2}} + \|\partial h\|_{D, \gamma}.$$

We therefore conclude, by using $\|\partial_z a^{(1)}\|_{\frac{1}{2}} \leq C$,

$$\begin{aligned} \|A(\partial_z a_\gamma^{(1)} - z_\gamma a_h^{(3)})\| &\leq C(\|\partial h\|_{D, \gamma} \\ &+ \|h\|_{D, \gamma - \frac{1}{2}} + \|\partial_z (a_\gamma^{(1)} - z_\gamma a_h^{(3)})\|)(\|a_h\| + \|\partial_z a_h\|). \end{aligned} \quad (3.30)$$

We then split $\|\partial_z (a_\gamma^{(1)} - z_\gamma a_h^{(3)})\|$ into

$$\|\langle \partial_z (a_\gamma^{(1)} - z_\gamma a_h^{(3)}), \tau \rangle \tau\| + \left\| \left(\partial_z (a_\gamma^{(1)} - z_\gamma a_h^{(3)}) \right)^\perp \right\|$$

By using (3.17), Theorem 3.2 and (3.20), the first term is bounded by $\|h\|_D$. The second can be absorbed in the left hand side for $\{\|a_h\| + \|\partial_z a_h\|\}$ small, by using the spectral gap for A . This concludes the proof of (3.23). \square

4. Energy Estimates and Decay.

In this section we obtain bounds on the L^2 -norms of the perturbation and its space and time derivatives, which will ensure the stability of the front solution, as well as on the γ -weighted norms which control the space decay of the perturbation and, as a consequence, the rate of convergence to zero of the perturbation as $t \rightarrow \infty$. All the estimates are obtained via an energy method based on a notion of “energy” which is constructed in terms of the linearization of the Liapunov functional \mathcal{G} , which replaces the usual entropy functional in the case of long range interactions. The so obtained energy involves the quadratic form associated to the operator A , as discussed in the Introduction. All the estimates are based on the following Lemma, depending on the structure of the linearized equation, which is common to the equation for the perturbation as well as to the one for its derivatives.

Lemma 4.1 Given $\Gamma = (\Gamma_1, \Gamma_2)$, let $g = (g_1, g_2)$ be the solution to the equation

$$\partial_t g_i + G_i g_i - L g_i = \Gamma_i, \quad (4.1)$$

with $G_i g_i$ defined in (1.9). Then, with the usual orthogonal decomposition

$$g = a_g M + (1 - P)g,$$

we have:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \int_{\mathbb{R}} dz a_g A a_g + \sum_{i=1}^2 \int_{\mathbb{R}} dz \int_{\mathbb{R}^3} dv \frac{1}{M w_i} |(I - P)g_i|^2 \right\} \\ & - \sum_{i=1}^2 \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{w_i M} (I - P)g_i L (I - P)g_i \\ & = \langle A a_g, \Gamma \rangle + \left\langle \frac{1}{M w} (I - P)g, \Gamma \right\rangle. \end{aligned}$$

Note that the inner product in right hand side is just the $L^2(dz dv)$ inner product.

Proof: Repeating the same computation as in Section 3, we have

$$M(\partial_t(a_g)_i + v_z w_i \partial_z(A a_g)_i) = -\partial_t(I - P)g_i - G_i(I - P)g_i + L(I - P)g_i + \Gamma_i.$$

We take the scalar product $(\cdot, \cdot)_M$ of (4.1) with $M w_i(A a_g)_i + (I - P)g_i$ to get:

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^2 \frac{d}{dt} \left[\int_{\mathbb{R} \times \mathbb{R}^3} dz dv M(a_g)_i(A a_g)_i + \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{M w_i} |(I - P)g_i|^2 \right] = \\ & - \sum_{i=1}^2 \left\{ \int_{\mathbb{R} \times \mathbb{R}^3} dz dv M w_i(A a_g)_i v_z \partial_z(A a_g)_i - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv (I - P)g_i v_z \partial_z(A a_g)_i \right. \\ & - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv (A a_g)_i G_i(I - P)g_i - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{M w_i} (I - P)g_i G_i(I - P)g_i \\ & \left. + \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{M w_i} (I - P)g_i L(I - P)g_i \right\} + \langle \Gamma, A a_g \rangle + \left\langle \frac{1}{M w} (I - P)g, \Gamma \right\rangle. \end{aligned}$$

The first term on the right hand side vanishes since $w_i(A a_g)_i \partial_z(A a_g)_i$ are functions of z, t only and the Maxwellian is centered. By recalling the definition of G_i , (1.9),

$$G_i(I - P)g_i = v_z \partial_z(I - P)g_i + U * w'_i \partial_{v_z}(I - P)g_i,$$

we have for the third term

$$\begin{aligned} - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv (A a_g)_i G_i(I - P)g_i &= - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv (A a_g)_i v_z \partial_z(I - P)g_i \\ &= \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \partial_z(A a_g)_i v_z (I - P)g_i \end{aligned}$$

which cancels with the second term $(-\int_{\mathbb{R} \times \mathbb{R}^3} dz dv (I - P)g_i v_z \partial_z (Aa_g)_i)$ in the right hand side. By using the definition of G_i we get for the fourth term

$$\begin{aligned} & - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{Mw_i} (I - P)g_i G_i (I - P)g_i \\ &= - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{1}{Mw_i} \frac{1}{2} [v_z \partial_z ((I - P)g_i)^2 + U * w'_j \partial_{v_z} ((I - P)g_i)^2] \\ &= - \int_{\mathbb{R} \times \mathbb{R}^3} dz dv \frac{v_z}{2Mw_i} \left[\frac{w'_i}{w_i} + \beta U * w'_j \right] ((I - P)g_i)^2 = 0 \end{aligned}$$

by using the equation for the front. \square

In the next Lemmas we apply above identity and the estimates in Section 3 to bound the weighted norms of h and its space and time derivatives.

Lemma 4.2 *Let $0 \leq \gamma$ be sufficiently small. Then if $\|h(t)\|_{M,\gamma} \leq \delta_0$*

$$\frac{1}{2} \frac{d}{dt} \|h(t)\|_{M,\gamma}^2 + \nu_0 \|h(t)\|_{D,\gamma}^2 \leq C(\gamma + \delta_0) (\|\partial_t h(t)\|_{D,\gamma}^2 + \|h(t)\|_{D,\gamma}^2), \quad (4.2)$$

with ν_0 given in Lemma 2.1.

Proof. Note that $g = z_\gamma h$ satisfies

$$\partial_t g_i + G_i g_i - Lg_i = F_i(h) \partial_{v_z} g_i + \hat{G}h_i + \frac{2zv_z \gamma g_i}{1 + z^2} \equiv \Gamma_i. \quad (4.3)$$

where

$$\hat{G}h_i = v_z M w_i \beta \int dz' U'(z - z') (z_\gamma - z'_\gamma) (a_h)_j(z', t). \quad (4.4)$$

We now apply Lemma 4.1. We first treat $F(a_h) \partial_{v_z} g$. Notice that

$$\langle F(a_h) \partial_{v_z} g, (Aa_g) \rangle = 0,$$

and

$$\begin{aligned} & \left| \sum_{i=1}^2 \left(F_i(h) \partial_{v_z} g_i, \frac{1}{Mw_i} (I - P)g_i \right) \right| \\ & \leq C \left[\|F(h)\|_{L^\infty} (\|a_g\| + \|\partial_{v_z} (I - P)g\|_M) \right] \|(I - P)g\|_M \\ & \leq C [\|\partial_t h\|_D + \|h\|_D] \|g\|_M \|g\|_D + \|h\|_M \|g\|_D^2 \\ & \leq C \delta_0 [\|h\|_D + \|\partial_t h\|_D]^2 + C \delta_0 \|g\|_D^2. \end{aligned}$$

Next we estimate $\hat{G}h_i$. Note that $\langle \hat{G}h, Aa_g \rangle = 0$. By (3.13),

$$|z_\gamma - z'_\gamma| \leq C\gamma |z - z'| (z_{\gamma - \frac{1}{2}} + z'_{\gamma - \frac{1}{2}}) \leq C\gamma |z - z'|$$

for $\gamma < \frac{1}{2}$, recalling $a_h = a_h^{(1)} + a_h^{(2)}$, we deduce that

$$\begin{aligned} & (\hat{G}h_i, \frac{1}{Mw_i}(I-P)g_i) \\ &= \beta \int dv v_z (I-P)g_i \left(\int dz' U \partial_z(z-z')[z_\gamma - z'_\gamma](a_h)_j(z', t) \right) \\ &= \beta \int dv v_z (I-P)g_i \left(\int dz' \partial_z U(z-z')[z_\gamma - z'_\gamma](a_h^{(2)})_j(z', t) \right) \\ & \quad + \beta \int dv v_z (I-P)g_i \left(\int dz' U(z-z')\partial_{z'}[(z_\gamma - z'_\gamma)(a_h^{(1)})_j(z', t)] \right), \end{aligned}$$

Hence

$$\begin{aligned} & \left| (\hat{G}h_i, \frac{1}{Mw_i}(I-P)g_i) \right| \\ & \leq C\gamma \left(\|a_h^{(2)}\| + \|\partial_z a_h^{(1)}\| + \|U * \frac{za_h^{(1)}}{(1+z^2)^{1-\gamma}}\| \right) \|(I-P)g\|_M \\ & \leq C\gamma [\|g\|_D^2 + \|\partial_t g\|_D^2]. \end{aligned}$$

We have used (3.12) and 3.9 in Lemma 3.3.

For the third term $\frac{2zv_z\gamma g_i}{1+z^2}$ in the definition of Γ_i , (4.3), since $v_z M^{-1} = \beta^{-1}\partial_{v_z} M^{-1}$, an integration by part in the v -variable and again estimate (3.12) give

$$\begin{aligned} & \sum_{i=1}^2 \left(\frac{2zv_z\gamma g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i \right) \\ &= \sum_{i=1}^2 \left(\frac{2zv_z\gamma(I-P)g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i \right) \\ & \quad + \sum_{i=1}^2 \left(\frac{2zv_z\gamma P g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i \right) \end{aligned}$$

Hence

$$\begin{aligned} & \left| \sum_{i=1}^2 \left(\frac{2zv_z\gamma g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i \right) \right| \\ & \leq C\gamma \sum_{i=1}^2 \left(\frac{2z(I-P)g_i}{1+z^2}, \frac{1}{\beta Mw_i} \partial_{v_z}(I-P)g_i \right) + C\gamma [\|\partial_t g\|_D^2 + \|g\|_D^2] \\ & \leq C\gamma [\|\partial_t g\|_D^2 + \|g\|_D^2]. \end{aligned}$$

On the other hand,

$$\left| \left\langle \frac{2zv_z\gamma(I-P)g}{1+z^2}, Aa_g \right\rangle \right| \leq C\gamma \|(I-P)g\|_M \left\| \frac{z}{1+z^2} Aa_g \right\|.$$

We use the splitting

$$\left\| \frac{z}{1+z^2} Aa_g \right\| \leq \left\| \frac{z}{1+z^2} Aa_g^{(2)} \right\| + \left\| \frac{z}{1+z^2} Aa_g^{(1)} \right\|$$

and introduce, as usual, the commutator to get

$$\left\| \frac{z}{1+z^2} Aa_g^{(2)} \right\| \leq \left\| \frac{zz_\gamma}{1+z^2} Aa^{(2)} \right\| + \left\| \frac{z}{1+z^2} [U, z_\gamma] a^{(2)} \right\|.$$

The above term is immediately bounded by using the definition of $Aa_h^{(2)}$, the second by using Theorem 3.2.

As for the contribution from $a_h^{(1)}$, we have

$$\begin{aligned} \left\| \frac{z}{1+z^2} Aa_g^{(1)} \right\| &\leq \left\| A \left(\frac{zz_\gamma}{1+z^2} a_h^{(1)} \right) \right\| + \left\| \left[A, \frac{z}{1+z^2} \right] z_\gamma a_h^{(1)} \right\| \\ &\leq C(\|\partial_t h\|_{D,\gamma} + \|h\|_{D,\gamma}). \end{aligned}$$

Indeed, the first term is bounded by using the boundedness of A and (3.12). To bound the commutator, we use

$$\left[\frac{z}{1+z^2}, A \right] z_\gamma a_h^{(1)} = \int_{\mathbb{R}} dz' U(z-z') \left(\frac{z}{1+z^2} - \frac{z'}{1+z'^2} \right) z'_\gamma a_h^{(1)}(z'), \quad (4.5)$$

the inequality

$$\left| \frac{z}{1+z^2} - \frac{z'}{1+z'^2} \right| \leq C \frac{|z-z'|}{(1+z^2)^{1/2}(1+z'^2)^{1/2}}, \quad (4.6)$$

and (3.10).

Therefore

$$\left| \left\langle \frac{2zv\gamma(I-P)g}{1+z^2}, Aa_g \right\rangle \right| \leq C\gamma \|g\|_D (\|\partial_t h\|_{D,\gamma} + \|h\|_{D,\gamma})$$

and this concludes the proof of the lemma. \square

We notice that in the proof of this Lemma we are allowed to apply (3.12) since we are assuming γ small enough and hence also $\gamma \leq \frac{1}{8}$.

Lemma 4.3 *If $\|\partial h(t)\|_{M,\gamma} + \|h(t)\|_M \leq \delta_0$, then*

$$\frac{1}{2} \frac{d}{dt} \|\partial_t h(t)\|_{M,\gamma}^2 + \nu_0 \|\partial_t h(t)\|_{D,\gamma}^2 \leq C(\gamma + \delta_0) \|\partial h(t)\|_{D,\gamma}^2 + C \|h(t)\|_{D,\gamma-\frac{1}{2}}^2. \quad (4.7)$$

Proof. Let $g = z_\gamma \partial_t h$. We have

$$[\partial_t + G_i - L]g_i = \frac{2zv_z \gamma g_i}{1+z^2} + \hat{G} \partial_t h_i + F_i(h) \partial_{v_z} g_i + F_i(\partial_t h) \partial_{v_z} h_i z_\gamma \equiv \Gamma_i. \quad (4.8)$$

By Lemma 4.1 we need to estimate

$$\langle Aa_g, \Gamma \rangle + \left\langle \frac{1}{Mw} (I - P)g, \Gamma \right\rangle$$

We first estimate the contribution due to $\frac{2\gamma zv_z g}{1+z^2}$. By using again $g = a_g M + (I - P)g$,

$$\left\langle \frac{2\gamma zv_z g}{1+z^2}, Aa_g \right\rangle = \left\langle \frac{2\gamma zv_z a_g M}{1+z^2}, Aa_g \right\rangle + \left\langle \frac{2\gamma zv_z (I - P)g}{1+z^2}, Aa_g \right\rangle.$$

By the same argument used in Lemma 4.2 we have

$$\begin{aligned} \left\| \frac{z}{1+z^2} Aa_g \right\| &\leq \left\| A \left(\frac{za_g}{1+z^2} \right) \right\| + C \left\| \int \frac{U(z-z')|z-z'|}{(1+z^2)^{1/2}(1+z'^2)^{1/2}} a_g(z') dz' \right\| \\ &\leq C \|(1+z^2)^{\gamma-\frac{1}{2}} \partial_t a_h\| \leq C \|(I - P) \partial_z h\|_{M, \gamma-\frac{1}{2}}. \end{aligned}$$

The last inequality is due to (3.4). We therefore have

$$\left| \left\langle \frac{2\gamma zv_z g}{1+z^2}, Aa_g \right\rangle \right| \leq \varepsilon \|\partial_t h\|_{D, \gamma}^2 + C_\varepsilon \gamma \|\partial_z h\|_{D, \gamma-\frac{1}{2}}^2.$$

We now estimate $\left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma g}{1+z^2} \right\rangle \right|$. As before, an integration by part in the v -variable, provides

$$\begin{aligned} &\left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma g}{1+z^2} \right\rangle \right| \\ &\leq \left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma (I - P)g}{1+z^2} \right\rangle \right| \\ &+ \left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma (1+z^2)^\gamma P \partial_t h}{1+z^2} \right\rangle \right| \quad (4.9) \\ &\leq C\gamma \left| \left\langle \frac{1}{\beta(1+z^2)^{1/2}} (I - P)g, \partial_{v_z} (I - P)g \right\rangle_M \right| \\ &+ C\gamma \|(I - P)g\|_M^2 + C_\varepsilon \|(1+z^2)^{\gamma-\frac{1}{2}} \partial_t a_h\|^2 \\ &\leq C\gamma \|\partial h\|_{D, \gamma-\frac{1}{2}}^2. \end{aligned}$$

We estimate the second term $\hat{G} \partial_t h_i$ in Γ by first noting that $\langle \hat{G} \partial_t h, (Aa_g) \rangle = 0$. With an argument similar to the one used to estimate $\hat{G}h$, we obtain

$$\left| \left\langle \frac{1}{Mw} (I - P)g, \hat{G} \partial_t h \right\rangle \right| \leq \varepsilon \|g\|_D^2 + C_\varepsilon \gamma \|\partial_t h\|_{D, \gamma-\frac{1}{2}}^2.$$

As for the third term in Γ , $F(h)\partial_{v_z}g$, we note that

$$\langle F(h)\partial_{v_z}g, (Aa_g) \rangle = 0,$$

and

$$\begin{aligned} & \left\langle F(h)\partial_{v_z}g, \frac{1}{Mw}(I-P)g \right\rangle \\ &= \left\langle F(h)\partial_{v_z}Pg, \frac{1}{Mw}(I-P)g \right\rangle + \left\langle F(h)\partial_{v_z}(I-P)g, \frac{1}{Mw}(I-P)g \right\rangle \\ &\leq C\|F_i(h)\|_\infty (\|a_g\| \cdot \|(I-P)g_i\|_M + \|\partial_{v_z}(I-P)g\|_M^2). \end{aligned}$$

Hence

$$\left| \left\langle F(h)\partial_{v_z}g, \frac{1}{Mw}(I-P)g \right\rangle \right| \leq C\|\partial_z a_h\| (\|a_g\| \cdot \|g\|_D + \|g\|_D^2).$$

To estimate the fourth term $z_\gamma F(\partial_t a_h)\partial_{v_z}h$, we first remind that

$$\langle z_\gamma F(\partial_t a_h)\partial_{v_z}h, Aa_g \rangle = 0.$$

Since by (3.4)

$$\|z_\gamma F(z_\gamma \partial_t a_h)\|_\infty \leq C\|(I-P)\partial_z h\|_{M,\gamma},$$

and

$$\|[z_\gamma, F]\partial_t a_h\|_\infty \leq C\|(I-P)\partial_z h\|_{M,\gamma-\frac{1}{2}},$$

we have, by using the smallness assumption and integrating by part on v ,

$$\begin{aligned} \left| \left\langle z_\gamma F(\partial_t a_h)\partial_{v_z}h, \frac{1}{Mw}(I-P)g \right\rangle \right| &\leq C\|h\|_M \|z_\gamma F(\partial_t a_h)\|_\infty \|\partial_t h\|_D \\ &\leq C\delta_0 \|\partial h\|_{D,\gamma}^2. \end{aligned}$$

This concludes the proof of the Lemma. \square

Lemma 4.4 *If $\|\partial h(t)\|_M + \|h(t)\|_M \leq \delta_0$, then*

$$\frac{1}{2} \frac{d}{dt} \|\partial_z h(t)\|_M^2 + \nu_0 \|\partial_z h(t)\|_D^2 \leq C (\|h\|_D^2 + \|\partial_t h\|_D^2 + \delta_0 \|\partial h\|_D^2) \quad (4.10)$$

Moreover, given $0 < \gamma \leq 1$, if $\|\partial h(t)\|_{M,\gamma} + \|h(t)\|_M \leq \delta_0$ and $\|\partial_z h\|_{M,\frac{1}{2}} < \eta$, then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_z h(t)\|_{M,\gamma}^2 + \nu_0 \|\partial_z h(t)\|_{D,\gamma}^2 \\ & \leq C \left(\|h\|_{D,\gamma-\frac{1}{2}}^2 + \|\partial_t h\|_{D,\gamma-\frac{1}{2}}^2 + \delta_0 \|\partial h\|_{D,\gamma}^2 + \gamma \|\partial_z h\|_{D,\gamma-\frac{1}{2}}^2 \right) \quad (4.11) \end{aligned}$$

Proof. We define $g = z_\gamma \partial_z h$ to get

$$\begin{aligned} \partial_t g_i + G_i g_i - L g_i &= \frac{2\gamma z v_z g_i}{1+z^2} + z_\gamma \beta v_z M w'_i U * \partial_z a_j \\ + \hat{G} \partial_z h_i + \partial_z U * w'_j \partial_{v_z}(z_\gamma h_i) - F_i(h) \partial_{v_z} g_i - z_\gamma F_i(\partial_z a_h) \partial_{v_z} h_i &\equiv \Gamma_i. \end{aligned}$$

where \hat{G} is defined in (4.4). Since $g = a_g M + (I - P)g$, by Lemma 4.1 we need to estimate

$$\langle \Gamma, A a_g \rangle + \left\langle \frac{1}{M w} (I - P)g, \Gamma \right\rangle.$$

In this proof, for consistency with the notation in Lemma 3.4, we switch from a_g to a_γ . We first estimate the first term $\frac{2\gamma z v_z g_i}{1+z^2}$.

$$\left\langle \frac{2\gamma z v_z g}{1+z^2}, A a_\gamma \right\rangle = \left\langle \frac{2\gamma z v_z a_\gamma M}{1+z^2}, A a_\gamma \right\rangle + \left\langle \frac{2\gamma z v_z (I - P)g}{1+z^2}, A a_\gamma \right\rangle.$$

The first contribution above vanishes. For the second term, we split $a_\gamma = a_\gamma^{(1)} + a_\gamma^{(2)}$. Then we have

$$\left\| \frac{z}{1+z^2} A a_\gamma^{(2)} \right\| \leq C \|\partial_z h\|_{D, \gamma - \frac{1}{2}}$$

by an argument similar to the one used in (3.24) – (3.27). On the other hand,

$$\frac{z}{1+z^2} A a_\gamma^{(1)} = A \left(a_\gamma^{(1)} \frac{z}{1+z^2} \right) + \left[\frac{z}{1+z^2}, A \right] a_\gamma^{(1)}.$$

By the boundedness of A ,

$$\left\| A \left(a_\gamma^{(1)} \frac{z}{1+z^2} \right) \right\| \leq C \|z_{\gamma - \frac{1}{2}} \partial_z a_h^{(1)}\|.$$

The commutator can be estimated in the usual way and we conclude that

$$\left\| \left[\frac{z}{1+z^2}, A \right] a_\gamma^{(1)} \right\| \leq C \|z_{\gamma - \frac{1}{2}} \partial_z a_h^{(1)}\|.$$

Collecting all the estimates and using Lemma 3.3 to bound $\|z_{\gamma - \frac{1}{2}} \partial_z a_h^{(1)}\|$, we have

$$\left\| \frac{z}{1+z^2} A a_\gamma^{(1)} \right\| \leq C \left(\|h\|_{D, \gamma - \frac{1}{2}} + \|\partial h\|_{D, \gamma - \frac{1}{2}} \right).$$

Therefore, for any $\varepsilon > 0$ there is C_ε such that

$$\left| \left\langle \frac{2\gamma z v_z (1 - P)g}{1+z^2}, A a_\gamma \right\rangle \right| \leq \varepsilon \|\partial_z h\|_{D, \gamma}^2 + C_\varepsilon \gamma (\|h\|_{D, \gamma - \frac{1}{2}}^2 + \|\partial h\|_{D, \gamma - \frac{1}{2}}^2).$$

We deal with next term as we already did to get (4.10): Using $v_z M^{-1} = \beta^{-1} \partial_{v_z} M^{-1}$, an integration by part in the v -variable, provides

$$\begin{aligned} & \left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma g}{1 + z^2} \right\rangle \right| \\ & \leq \left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma (I - P)g}{1 + z^2} \right\rangle \right| + \left| \left\langle \frac{1}{Mw} (I - P)g, \frac{zv_z \gamma z_\gamma P \partial_z h}{1 + z^2} \right\rangle \right| \\ & \leq C\gamma \left\| \frac{1}{1 + |z|} (I - P)g \partial_{v_z} (I - P)g \right\|_M + \varepsilon \|(I - P)g\|_M^2 + C_\varepsilon \gamma \|z_{\gamma - \frac{1}{2}} \partial_z a_h\|^2 \\ & \leq \varepsilon \|(I - P)g\|_M^2 + C_\varepsilon \gamma \left(\|h\|_{D, \gamma - \frac{1}{2}}^2 + \|\partial h\|_{D, \gamma - \frac{1}{2}}^2 + \|h\|_D^2 + \|\partial h\|_D^2 \right). \end{aligned}$$

In the last inequality we have used Lemma 3.3 and Lemma 3.4.

The second term is easily obtained by Theorem 3.2. Indeed,

$$\langle z_\gamma \beta v_z M w'_i U * \partial_z a_j, (Aa_\gamma)_i \rangle = 0$$

and

$$\begin{aligned} & \left| \langle z_\gamma \beta v_z \frac{w'_i}{w} U * \partial_z a_j, (I - P)g_i \rangle \right| \\ & \leq \varepsilon \|(I - P)g\|_M^2 + C_\varepsilon \left(\|\partial_z a_h^{(1)}\|^2 + \|a_h^{(2)}\|^2 \right) \\ & \leq \varepsilon \|(I - P)g\|_M^2 + C_\varepsilon \left(\|\partial_t h\|_D^2 + \|h\|_D^2 \right). \end{aligned}$$

We now estimate the fourth term $z_\gamma \partial_z U * w'_j \partial_{v_z} h_i$. Notice that

$$\langle z_\gamma \partial_z U * w' \partial_{v_z} Ph, (Aa_\gamma) \rangle = 0.$$

Introducing as usual the commutator, we obtain

$$\begin{aligned} |\langle z_\gamma \partial_z U * w' \partial_{v_z} (1 - P)h, Aa_\gamma \rangle| & \leq |\langle z_\gamma \partial_z U * w' \partial_{v_z} (1 - P)h, z_\gamma A \partial_z a_h \rangle| \\ & \quad + |\langle z_\gamma \partial_z U * w' \partial_{v_z} (1 - P)h, [A, z_\gamma] \partial_z a_h \rangle|. \end{aligned}$$

Then we have the bound $|z_\gamma^2 w'| \leq C$ because of the fast decay of w' and $\|A \partial_z a_h\| \leq C(\|h\|_D + \|\partial h\|_D)$ by using (3.21) and (3.11). The commutator term is bounded by using (3.14) to obtain $\|[A, z_\gamma] \partial_z a_h\| \leq C \|\partial_z a_h\|_{\gamma - \frac{1}{2}}$ which is then bounded again by using (3.21) and (3.11). In conclusion we obtain:

$$|\langle z_\gamma \partial_z U * w' \partial_{v_z} (1 - P)h, Aa_\gamma \rangle| \leq C \left(\|h\|_{D, \gamma - \frac{1}{2}} + \|\partial h\|_{D, \gamma - \frac{1}{2}} \right) \|h\|_D.$$

On the other hand,

$$\begin{aligned} & \langle z_\gamma \partial_z U * w' \partial_{v_z} h, \frac{1}{Mw} (I - P)g \rangle \\ & = \langle z_\gamma \partial_z U * w' \partial_{v_z} (I - P)h, \frac{1}{Mw} (I - P)g \rangle \\ & \quad + \langle z_\gamma \partial_z U * w' \partial_{v_z} Ph, \frac{1}{Mw} (I - P)g \rangle. \end{aligned}$$

The first term is clearly bounded by $\varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\|h\|_D^2$. For the second term,

$$\begin{aligned} & \left| \sum_{i=1}^2 (z_\gamma \partial_z U * w'_j \partial_{v_z} P h, \frac{1}{Mw} (I-P)g) \right| \\ & \leq \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon \left(\|\partial_t h\|_D^2 + \|h\|_D^2 + \|a_h^{(2)}\|^2 + \|h\|_D^2 \right). \end{aligned}$$

To get the last inequality we have decomposed, as usual $a_h = a_h^{(1)} + a_h^{(2)}$ and bounded the contribution due to $a_h^{(2)}$ by the first part of Theorem 3.2 and the one due to $a_h^{(1)}$ by using (3.10) and the fast decay of w' . Then, by using Theorem 3.2 we get the final bound

$$\begin{aligned} & \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon \left(\|\partial_z a_h^{(1)}\|^2 + \|a_h^{(2)}\|^2 + \|h\|_D^2 \right) \\ & \leq \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon \left(\|\partial_t h\|_D^2 + \|h\|_D^2 \right). \end{aligned}$$

Now turn to the third term $\hat{G}h$ in Γ . Since $\langle \hat{G}h, Aa_\gamma \rangle = 0$,

$$\begin{aligned} & \left| \left\langle \frac{1}{Mw} (I-P)g, \hat{G} \partial_z a_h \right\rangle \right| \\ & \leq \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon \|\partial_t h\|_{D, \gamma - \frac{1}{2}}^2 + C_\varepsilon \|h\|_{D, \gamma - \frac{1}{2}}^2. \end{aligned}$$

For the fifth term $F(a_h) \partial_{v_z} g$, we note that $\langle F(a_h) \partial_{v_z} g, Aa_\gamma \rangle = 0$ and

$$\begin{aligned} & \left| \left\langle F(a_h) \partial_{v_z} g, \frac{1}{Mw} (I-P)g \right\rangle \right| \\ & \leq \left| \left\langle F(a_h) \partial_{v_z} P g, \frac{1}{Mw} (I-P)g \right\rangle \right| + \left| \left\langle F(a_h) \partial_{v_z} (I-P)g, \frac{1}{Mw} (I-P)g \right\rangle \right| \\ & \leq C \|F(a_h)\|_\infty (\|a_\gamma\| \|(I-P)g\|_M + \|\partial_{v_z} (I-P)g\|_M^2) \\ & \leq C \|\partial_z a_h\| (\|a_\gamma\| \|(I-P)g_i\|_M + \|\partial_{v_z} (I-P)g\|_M^2) \\ & \leq C \delta_0 (\|g\|_D^2 + \|h\|_D^2 + \|\partial_t h\|_D^2). \end{aligned}$$

Finally, to estimate the sixth term $z_\gamma F(\partial_z a_h) \partial_{v_z} h$, we note

$$\int_{\mathbb{R}^3} dv z_\gamma F_i(\partial_z a_h) \partial_{v_z} h_i (Aa_\gamma)_i = 0.$$

To treat the last term we consider separately the case $\gamma = 0$ and the case $\gamma > 0$. In the first case we simply get,

$$\begin{aligned} & \left| \left\langle F(\partial_z a_h) \partial_{v_z} h, \frac{1}{Mw} (I-P) \partial_z h \right\rangle \right| \\ & \leq C (\|\partial_z a_h\| \|a_h\| + \|\partial_z a_h\| \|h\|_D) \|(I-P) \partial_z h\|_M \\ & \leq C \delta_0 \|\partial h\|_D^2 \end{aligned}$$

by using (3.11) and (3.21) with $\gamma = 0$ to bound $\|\partial_z a_h\|$.

In the case $\gamma > 0$ we need to employ Lemma 3.4 to treat the last term as

$$\begin{aligned}
& \left| \left\langle z_\gamma F(\partial_z a_h) \partial_{v_z} h, \frac{1}{Mw} (I - P)g \right\rangle \right| \\
& \leq \left| \left\langle F(a_\gamma) \partial_{v_z} h, \frac{1}{Mw} (I - P)g \right\rangle \right| + \left| \left\langle ([z_\gamma, F](\partial_z a_h) \partial_{v_z} h), \frac{1}{Mw} (I - P)g \right\rangle \right| \\
& \leq \left(\|a_\gamma^{(2)}\| \cdot \|a_h\| + \|a_\gamma\| \cdot \|h\|_D \right) \|(I - P)g\|_M \\
& \quad + \left((\|\partial_z \{a_\gamma^{(1)} - z_\gamma a_h^{(3)}\}\| + \|z_\gamma a_h^{(3)}\|) \cdot \|a_h\| \right) \|(I - P)g\|_M \\
& \quad + \left(\|z_{\gamma - \frac{1}{2}} \partial_z a_h^{(1)}\| \cdot \|a_h\| \right) \|(I - P)g\|_M \\
& \leq C\delta_0 \left(\|g\|_D^2 + \|h\|_D^2 + \|\partial h\|_{D,\gamma}^2 + \|h\|_{D,\gamma - \frac{1}{2}}^2 + \|\partial_t h\|_{D,\gamma - \frac{1}{2}}^2 \right).
\end{aligned}$$

We deduce our lemma by letting ε small and using δ_0 small. \square

We remark that this is the only point where we use (3.23). The relevance of this estimate is in the fact that we get a bound involving the norm of the function with a power $\gamma - \frac{1}{2}$. This is crucial for the final consistency argument.

Proof of Theorem 1.2: To prove the first part, we start with $\gamma = 0$ in all three Lemmas 4.2, 4.3 and 4.4. We multiply by a positive number K (4.2) and (4.7) and add both to (4.10):

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(K (\|h(t)\|_M^2 + \|\partial_t h(t)\|_M^2) + \|\partial_z h(t)\|_M^2 \right) \\
& + K\nu_0 \left(\|h(t)\|_D^2 + \|\partial_t h(t)\|_D^2 \right) + \nu_0 \|\partial_z h(t)\|_D^2 \\
& \leq C \left(K\delta_0 (\|h(t)\|_D^2 + \|\partial h(t)\|_D^2) + \|h(t)\|_D^2 \right. \\
& \quad \left. + \|\partial_t h(t)\|_D^2 + \delta_0 \|\partial_z h(t)\|_D^2 \right).
\end{aligned}$$

By choosing $K > \frac{C}{4\nu_0}$, and $\delta_0 < \frac{\nu_0}{4C}$, we obtain that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(K (\|\partial_t h\|_M^2 + \|h\|_M^2) + \|\partial_z h\|_M^2 \right) \\
& + \frac{\nu_0}{2} \left(K (\|\partial_t h\|_D^2 + \|h\|_D^2) + \|\partial_z h\|_D^2 \right) \leq 0. \tag{4.12}
\end{aligned}$$

Then, a standard continuity argument shows that the assumption

$$\|h(t)\|_M^2 + \|\partial h(t)\|_M^2 \leq \delta_0$$

is verified at any time t , thus completing the proof of the first part of Theorem 1.2. In particular, we have

$$\int_0^\infty dt \left(K (\|\partial_t h(t)\|_D^2 + \|h(t)\|_D^2) + \|\partial_z h(t)\|_D^2 \right) \leq C (\|h(0)\|_M^2 + \|\partial h(0)\|_M^2). \tag{4.13}$$

To prove the second part, we first prove an inequality like (4.12) for the weighted norms with weight z_{γ_0} , for γ_0 small. We will use a standard continuity argument with the assumption

$$(\|h(t)\|_{M,\gamma_0}^2 + \|\partial h(t)\|_{M,\frac{1}{2}+\gamma_0}^2) \leq \delta_0. \quad (4.14)$$

As first step, we multiply once again (4.2) and (4.7) by K and add them to (4.11)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(K(\|h(t)\|_{M,\gamma_0}^2 + \|\partial_t h(t)\|_{M,\gamma_0}^2) + \|\partial_z h(t)\|_{M,\gamma_0}^2 \right) \\ & + K\nu_0 \left(\|h(t)\|_{D,\gamma_0}^2 + \|\partial_t h(t)\|_{D,\gamma_0}^2 \right) + \nu_0 \|\partial_z h(t)\|_{D,\gamma_0}^2 \\ & \leq KC(\gamma_0 + \delta_0) \left(\|\partial_t h(t)\|_{D,\gamma_0}^2 + \|h(t)\|_{D,\gamma_0}^2 + \|\partial h(t)\|_{D,\gamma_0-\frac{1}{2}}^2 \right) \\ & + KC\|h(t)\|_{D,\gamma_0-\frac{1}{2}}^2 + C \left(\|h(t)\|_{D,\gamma_0-\frac{1}{2}}^2 \right. \\ & \left. + \|\partial_t h(t)\|_{D,\gamma_0-\frac{1}{2}}^2 + \gamma_0 \|\partial_z h(t)\|_{D,\gamma_0-\frac{1}{2}}^2 + \delta_0 \|\partial h(t)\|_{D,\gamma_0}^2 \right). \end{aligned}$$

For γ_0 small enough, for δ_0 small enough and K sufficiently large we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(K(\|h(t)\|_{M,\gamma_0}^2 + \|\partial_t h(t)\|_{M,\gamma_0}^2) + \|\partial_z h(t)\|_{M,\gamma_0}^2 \right) \\ & + K\nu_0 \left(\|h(t)\|_{D,\gamma_0}^2 + \|\partial_t h(t)\|_{D,\gamma_0}^2 \right) + \nu_0 \|\partial_z h(t)\|_{D,\gamma_0}^2 \\ & \leq KC(\|\partial h(t)\|_D^2 + \|h(t)\|_D^2). \end{aligned}$$

Then, as before, by using (4.13) we can conclude that

$$\begin{aligned} & \nu_0 \int_0^\infty dt \left(K\nu_0 (\|h(t)\|_{D,\gamma_0}^2 + \|\partial_t h(t)\|_{D,\gamma_0}^2) + \nu_0 \|\partial_z h(t)\|_{D,\gamma_0}^2 \right) \\ & \leq C(\|h(0)\|_{M,\gamma_0}^2 + \|\partial h(0)\|_{M,\gamma_0}^2). \end{aligned} \quad (4.15)$$

Finally, we let $\gamma = \gamma_0$ sufficiently small in (4.2) of Lemma 4.2, while let $\gamma = \frac{1}{2} + \gamma_0$ in (4.7) and (4.10) of Lemmas 4.3 and 4.4. We then multiply a large constant K to both (4.2) and (4.7) and add them to (4.10) to get:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(K \left(\|h(t)\|_{M,\gamma_0}^2 + \|\partial_t h(t)\|_{M,\gamma_0+\frac{1}{2}}^2 \right) + \|\partial_z h(t)\|_{M,\gamma_0+\frac{1}{2}}^2 \right) \\ & + K \left(\nu_0 \|h(t)\|_{D,\gamma_0}^2 + \nu_0 \|\partial_t h(t)\|_{D,\gamma_0+\frac{1}{2}}^2 \right) + \nu_0 \|\partial_z h(t)\|_{D,\gamma_0+\frac{1}{2}}^2 \\ & \leq KC(\gamma_0 + \frac{1}{2} + \delta_0) (\|\partial h(t)\|_{D,\gamma_0}^2 + \|h(t)\|_{D,\gamma_0}^2) \\ & + C \left(\frac{1}{2} \|\partial_z h(t)\|_{D,\gamma_0}^2 + \|h(t)\|_{D,\gamma_0}^2 + \|\partial_t h(t)\|_{D,\gamma_0}^2 + \delta_0 \|\partial h(t)\|_{D,\gamma_0+\frac{1}{2}}^2 \right), \end{aligned}$$

Then, there is a large constant K such that, for δ_0 small,

$$\begin{aligned} & \frac{d}{dt} \left(K(\|\partial_t h\|_{M, \frac{1}{2} + \gamma_0}^2 + K\|h\|_{M, \gamma_0}^2) + \|\partial_z h\|_{M, \frac{1}{2} + \gamma_0} \right) \\ & + \nu_0 K \left(\|\partial_t h\|_{D, \frac{1}{2} + \gamma_0}^2 + \|h\|_{D, \gamma_0}^2 \right) + \nu_0 \|\partial_z h\|_{D, \gamma_0 + \frac{1}{2}} \\ & \leq KC \left(\|\partial h\|_{D, \gamma_0}^2 + \|h\|_{D, \gamma_0}^2 \right). \end{aligned}$$

Using (4.15) and a standard continuity argument, we obtain:

$$\begin{aligned} & \sup_{0 \leq t \leq \infty} \left(\|h(t)\|_{M, \gamma_0} + \|\partial h(t)\|_{M, \frac{1}{2} + \gamma_0} \right) \\ & \leq C \left(\|h(0)\|_{M, \gamma_0} + \|\partial h(0)\|_{M, \frac{1}{2} + \gamma_0} \right) \end{aligned} \quad (4.16)$$

and the a priori assumption (4.14) is valid when $\|h(0)\|_{M, \gamma_0} + \|\partial h(0)\|_{M, \frac{1}{2} + \gamma_0}$ is sufficiently small.

We now turn back to (4.12). We want to control $\|h\|_M + \|\partial h\|_M$ but up to now we only have a uniform bound on $\|h\|_D + \|\partial h\|_D$. What is missing is a bound on $\|a_h\|$. But from (4.16) and an interpolation,

$$\begin{aligned} \|a_h^{(1)}\| & \leq \|(1+z^2)^{1/2} \partial_z a_h^{(1)}\| \\ & \leq C \|(1+z^2)^{\frac{1}{2} + \gamma_0} \partial_z a_h^{(1)}\|_{\frac{1}{1+2\gamma_0}} \times \|\partial_z a_h^{(1)}\|_{\frac{2\gamma_0}{1+2\gamma_0}} \\ & \leq C \{ \|h(0)\|_{\gamma_0} + \|\partial h(0)\|_{\frac{1}{2} + \gamma_0} \}^{\frac{1}{1+2\gamma_0}} \|h\|_D^{\frac{2\gamma_0}{1+2\gamma_0}}. \end{aligned}$$

As for $a_h^{(2)}$, and ∂a_h , by Lemma 3.2, we conclude that they satisfy the same inequality above with $\gamma_0 = 0$. Therefore, let $E_{\gamma_0} = \{ \|h(0)\|_{\gamma_0}^2 + \|\partial h(0)\|_{\frac{1}{2} + \gamma_0}^2 \}$

$$\{ \|h\|_D^2 + \|\partial h\|_D^2 \} \geq CE_0^{-\frac{1}{2\gamma_0}} \{ \|h\|^2 + \|\partial h\|^2 \}^{\frac{1+2\gamma_0}{2\gamma_0}}.$$

We thus conclude that:

$$\begin{aligned} & \frac{d}{dt} \{ \|\partial_z h\|_M^2 + K(\|h\|_M^2 + \|\partial_t h\|_M^2) \} \\ & + CE_0^{-\frac{1}{2\gamma_0}} \{ \|\partial_z h\|_M^2 + K(\|h\|_M^2 + \|\partial_t h\|_M^2) \}^{1+\frac{1}{2\gamma_0}} \leq 0. \end{aligned}$$

Denoting $y(t) \equiv \|\partial_z h\|_M^2 + K(\|h\|_M^2 + \|\partial_t h\|_M^2)$, we have

$$y' y^{-1-\frac{1}{2\gamma_0}} \leq -CE_0^{-\frac{1}{2\gamma_0}}.$$

By integrating between 0 and t , we deduce

$$\frac{1}{2\gamma_0} \{y(0)\}^{-\frac{1}{2\gamma_0}} - \frac{1}{2\gamma_0} \{y(t)\}^{-\frac{1}{2\gamma_0}} \leq -CE_0^{-\frac{1}{2\gamma_0}} t.$$

Hence from $y(0) \leq E_0$ we obtain

$$\begin{aligned} \frac{1}{2\gamma_0} \{y(t)\}^{-\frac{1}{2\gamma_0}} & \geq t \frac{C}{2\gamma_0} E_0^{-\frac{1}{2\gamma_0}} + \{y(0)\}^{-\frac{1}{2\gamma_0}} \\ & \geq \{ t \frac{C}{2\gamma_0} + 1 \} E_0^{-\frac{1}{2\gamma_0}} \end{aligned}$$

and the proof is completed by solving for $y(t)$. \square

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