Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

GHOST EFFECT BY CURVATURE IN PLANAR COUETTE FLOW

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(Communicated by the associate editor name)

To the memory of Carlo Cercignani

Abstract. We study a rarefied gas, described by the Boltzmann equation, between two coaxial rotating cylinders in the small Knudsen number regime. When the radius of the inner cylinder is suitably sent to infinity, the limiting evolution is expected to converge to a modified Couette flow which keeps memory of the vanishing curvature of the cylinders (*ghost effect* [17]). In the 1-d stationary case we prove the existence of a positive isolated L_2-solution to the Boltzmann equation and its convergence. This is obtained by means of a truncated bulk-boundary layer expansion which requires the study of a new Milne problem, and an estimate of the remainder based on a generalized spectral inequality.

1. **Introduction.** It is well known (see for example [9, 17, 14] and references quoted therein) that the asymptotic behavior of the solutions to the Boltzmann equation, in the limit of small Knudsen numbers, is well approximated by the compressible Euler equation for a perfect gas, while the viscosity and heat conducting effects are seen as first order corrections in the Knudsen number. To get finite size viscosity effect is more delicate because of the von Karman relation [20] between the Reynolds, Knudsen and Mach numbers, Re, Kn, Ma:

$$Ma \sim ReKn.$$

When Kn is small, either Re^{-1} or Ma have to be small. Therefore, finite viscosity effects are attained only if one assumes that the Mach number is of the same order as the Knudsen number.

With the extra assumption that density and temperature profiles differ from constants at most for terms of the order of the Knudsen number, it is then possible to show that the asymptotic behavior of the solutions to the Boltzmann equation

2000 *Mathematics Subject Classification.* Primary: 82B40, 82C26; Secondary 76P05.

Key words and phrases. Boltzmann equation, Hydrodynamical limit, Ghost effect, Couette flow.

are well approximated by the incompressible Navier-Stokes equations (INS), in the sense that the average velocity field, rescaled by the Mach number, converges to a solution u to INS. Moreover, the first order correction to the temperature profile converges to a solution to the heat equation with a convective term due to the rescaled velocity field u. Such results have been proved in several papers. An overview is provided in [14], to which we refer for a partial list of references on the subject. We also stress that the asymptotic behavior of the solutions to the compressible Navier-Stokes-Fourier equations, in the low Mach number limit and with the same assumption on density and temperature, is the same.

When the density and temperature do not satisfy the above mentioned assumptions, the Boltzmann equation deviates from the compressible Navier-Stokes-Fourier equations. Such a discrepancy, called *ghost effect* [17], is the issue we want to address in this paper.

The name is suggested by the fact that a small velocity field produces finite size modifications of the usual heat equation. This fact is confirmed both numerically and experimentally and devices as a vacuum pump have been designed starting from this effect. There has been a big theoretical, numerical and experimental work on this and for the details we refer to [17, 18]. We remark that the time dependent equations on the torus, in the diffusive space-time scaling were written in [11] and analyzed in [7], where it is observed that similar equations had been previously considered in some special cases [15]. Very little is known from the mathematical point of view, and the only rigorous result we are aware of is [8]. By using the techniques illustrated in the present paper, it is possible to deal with the time dependent problem in the torus, but that will be the subject of future works.

In this paper instead, we want to study a different type of ghost effect, pointed out in [19, 17] as *ghost effect by curvature*. It consists in the following: if one looks at the Couette flow between two coaxial rotating cylinders in the limit when the radius of the inner cylinder goes to infinity, one expects to obtain, as asymptotic behavior, the standard planar Couette flow. The analysis based on the Boltzmann equation does not confirm this. Indeed, an extra term appears in the limiting equations, which is reminiscent of the original structure of the problem. Hence, an infinitesimal curvature produces a finite size discrepancy, the ghost effect by curvature. The extra term gives rise to a bifurcation of the laminar stationary solution, which is absent in the standard Couette flow.

To be more specific, we look at the behavior of a rarefied gas between two coaxial cylinders of radius L and $L+1$, described by the Boltzmann equation in the diffusive space-time scaling. In cylindrical coordinates $(r, \theta, z) \in (L, L+1) \times [0, 2\pi) \times \mathbb{R}$ it is written as

$$\varepsilon \frac{\partial F}{\partial t} + v_r \frac{\partial F}{\partial r} + \frac{v_\theta}{r} \frac{\partial F}{\partial \theta} + v_z \frac{\partial F}{\partial z} + \frac{v_\theta}{r} \left(v_\theta \frac{\partial F}{\partial v_r} - v_r \frac{\partial F}{\partial v_\theta} \right) = \frac{1}{\varepsilon} Q(F, F), \tag{1.1}$$

where the positive and normalized function $F(r, \theta, z, v_r, v_\theta, v_z, t)$ is the probability density of particles in cylindrical coordinates and we have denoted by ε the Knudsen number. (v_r, v_θ, v_z) are the components of the velocity in the local basis associated to the cylindrical coordinates and $Q(f, g)$ is the Boltzmann collision integral for hard spheres:

$$Q(f, g)(v) = \frac{1}{2} \int_{\mathbb{R}^3} dv_* \int_{S_2} dn B(n, v - v_*) \big\{ f'_* g' + f' g'_* - f_* g - g_* f \big\}, \tag{1.2}$$

with f', f'_*, f, f_* standing for $f(v'), f(v'_*), f(v), f(v_*)$ respectively, $S_2 = \{n \in \mathbb{R}^3 \,|n^2 = 1\}$, B is the differential cross section $2B(n,V) = |V \cdot n|$ corresponding to hard spheres, and v, v_* and v',v'_* are post-collisional and precollisional velocities in an elastic collision:

$$v' = v - n(v - v_*) \cdot n, \quad v'_* = v_* + n(v - v_*) \cdot n. \tag{1.3}$$

The use of cylindrical coordinates produces a force-like term depending on the velocity, which will be referred below as *centrifugal force*.

We will look at the above equation in the planar limit, where one takes the radius of the inner cylinder L to infinity. A convenient change of variables is the following:

$$r = L + \frac{y+\pi}{2\pi}; \quad L\theta = -x; \quad v_r = v_y; \quad v_\theta = -v_x,$$

while the z variable is unchanged. For simplicity, we assume the distribution F invariant under rotations around the axis of the cylinders, so we drop the dependence on θ. Moreover, we scale $1/L$ proportionally to the inverse of the square of the Knudsen number:

$$\frac{1}{L} = \frac{\varepsilon^2}{c^2}, \tag{1.4}$$

with a constant c, related to the curvature, which will be specified below. With these assumptions the equation becomes

$$\varepsilon \frac{\partial F}{\partial t} + \; v_y \frac{\partial F}{\partial y} + v_z \frac{\partial F}{\partial z} + \frac{\varepsilon^2}{c^2} \sigma(y) v_x \left(v_x \frac{\partial F}{\partial v_y} - v_y \frac{\partial F}{\partial v_x} \right) = \frac{1}{\varepsilon} Q(F,F), \tag{1.5}$$

with

$$\sigma(y) = \frac{2\pi}{2\pi + \frac{\varepsilon^2}{c^2}(y+\pi)} \; . \tag{1.6}$$

The variable y varies between $-\pi$, corresponding to the inner cylinder, and π corresponding to the outer cylinder.

The boundary conditions on the two cylinders are assumed to be given by the diffuse reflection condition, meaning that the distribution of the incoming particles is Maxwellian:

$$\begin{aligned} F(-\pi, z, v, t) &= \alpha_-(F) \tilde{M}_-, \quad v_y > 0, \\ F(\pi, z, v, t) &= \alpha_+(F) \tilde{M}_+, \quad v_y < 0. \end{aligned} \tag{1.7}$$

We use the following notation: for $\rho > 0$, $T > 0$ and $u \in \mathbb{R}^3$

$$M(\rho, T, u; v) = \frac{\rho}{(2\pi T)^{3/2}} \mathrm{e}^{-\dfrac{|v-u|^2}{2T}} \tag{1.8}$$

is the Maxwellian with density ρ, temperature T and mean velocity u. In this paper we consider the two cylinders at the same temperature $T = 1$ and rotating with velocities U_- and U_+. Therefore

$$\tilde{M}_\pm = M(\sqrt{2\pi}, 1, (U_\pm, 0, 0); v), \tag{1.9}$$

where the density has been fixed so that the normalization condition

$$\int_{v_y \gtrless 0} dv |v_y| \tilde{M}_\pm = 1 \tag{1.10}$$

is satisfied. The constants $\alpha_\pm$, depending on the outgoing flow at the boundaries, are determined by the condition of vanishing net flow in the radial direction:

$$\int_{\mathbb{R}^3} dv v_y F(y, z, v_x, v_y, v_z, t) = 0, \quad \text{for } y = -\pi, \pi. \tag{1.11}$$

By using (1.7) and (1.10), one immediately gets

$$\begin{aligned} \alpha_-(F) &= -\int_{v_y<0} dv v_y F(-\pi, z, v_x, v_y, v_z, t), \\ \alpha_+(F) &= \int_{v_y>0} dv v_y F(\pi, z, v_x, v_y, v_z, t). \end{aligned} \tag{1.12}$$

We need to introduce a low Mach number assumption. Due to the particular geometry we consider here, we do not need that the full velocity field is small. Indeed the tangential component can be of order 1, while we need the radial and axial components to be of the order of the Knudsen number ε. We will use the notation $\hat{v}$ to denote the couple (v_y, v_z). Correspondingly, $\hat{\nabla} = (\partial_y, \partial_z)$. The Mach number assumption is therefore:

$$\hat{u} := \int_{\mathbb{R}^3} dv \hat{v} F = \mathcal{O}(\varepsilon). \tag{1.13}$$

The tangential component of the velocity, denoted by U is $\mathcal{O}(1)$ with respect to the Knudsen number ε. However, we will also need some smallness assumption on it. Therefore, we introduce another parameter δ, measuring the size of U:

$$U := \int_{\mathbb{R}^3} dv v_x F = \mathcal{O}(\delta). \tag{1.14}$$

The fact that δ, although small, will be chosen much larger than ε, is responsible of the arising of a ghost effect. In principle δ might be completely independent of ε, but, for technical reasons, we will assume, in the estimate of the remainder, a specific relation between δ and ϵ. Therefore, from now on, we replace U with δU with $U = \mathcal{O}(1)$ both in ε and δ. In order to get equations without singular terms when $\delta \to 0$, we will also assume the constant c appearing in the definition of σ (1.6) of order δ:

$$c = \delta C \tag{1.15}$$

for some other constant C also of order 1.

As usual we will look for a solution to (1.5) in terms of a truncated expansion in ε. The collision term forces the lowest order of the expansion to be a local Maxwellian. In order to fulfil the low Mach number assumptions (1.13), (1.14) and the boundary conditions (1.7), the lowest order has to be of the form

$$M_\delta = M(1 + \delta r, 1 + \delta\tau, (\delta U, 0, 0); v) = \frac{1 + \delta r}{(2\pi(1 + \delta\tau))^{3/2}} \exp\left[-\frac{\bar{v}^2}{2(1 + \delta\tau)}\right], \tag{1.16}$$

where

$$\bar{v} = v - (\delta U, 0, 0) = (v_x - \delta U, \hat{v}). \tag{1.17}$$

Note that the functions r and τ, representing corrections of order δ to the density and temperature, vanish at the boundary because we are restricting ourselves to the case when the two cylinders are at the same temperature. On the other hand gradients of U of order δ warm up the fluid in the bulk and may produce variations of the temperature and density of order δ^2.

The solution is sought for in the form

$$F = M_\delta + \Phi + \varepsilon \mathcal{R}, \tag{1.18}$$

where

$$\Phi = \sum_{n=1}^{N} \varepsilon^n F_n \tag{1.19}$$

for a suitable choice of N, and $\mathcal{R}$ is a remainder. In the next section we will give the procedure to compute the functions F_n's, which will be based on a kind of Hilbert expansion for the bulk parts B_n of F_n and a boundary layer expansion in order to restore the boundary conditions violated by the bulk terms. The computation of the bulk terms requires the solution of a rather complex system of equations for the hydrodynamical fields U, $\hat{u}$, r, τ depending on δ. They are

$$\begin{aligned}
&\hat{\nabla}[r+\tau] + \delta\hat{\nabla}[r\tau] = 0,\\
&\hat{\nabla}\cdot\hat{u} + \delta(\partial_t r + \hat{\nabla}\cdot(\hat{u}r)) = 0,\\
&(1+\delta r)\left(\partial_t U + \hat{u}\cdot\hat{\nabla}U\right) = \eta_0\hat{\Delta}U + \hat{\nabla}\cdot(\eta_\delta\hat{\nabla}U),\\
&(1+\delta r)\big(\partial_t\hat{u} + \hat{u}\cdot\hat{\nabla}\hat{u}\big) + \hat{\nabla}\mathcal{P}_2 - \frac{1}{C^2}\rho U^2 e_y = \eta_0\hat{\Delta}\hat{u}\\
&+\hat{\nabla}\cdot\left(\eta_\delta\hat{\nabla}\hat{u} + \frac{\delta^2}{\mathcal{P}}\big[\sigma_1\hat{\nabla}\tau\otimes\hat{\nabla}\tau + \sigma_2\hat{\nabla}U\otimes\hat{\nabla}U\big]\right),\\
&\frac{3}{2}(1+\delta r)\partial_t\tau + \frac{5}{2}(1+\delta r)\hat{u}\cdot\hat{\nabla}\tau = \kappa_0\hat{\Delta}\tau + \hat{\nabla}(\kappa_\delta\hat{\nabla}\tau) + \delta\eta|\hat{\nabla}U|^2.
\end{aligned} \tag{1.20}$$

Here η, κ, σ_1 and σ_2 are suitable transport coefficients depending on $1+\delta\tau$. We have set $\eta = \eta_0 + \eta_\delta$, $\kappa = \kappa_0 + \kappa_\delta$, with η_0 and κ_0 the values corresponding to $\delta = 0$ and η_δ, κ_δ the differences. e_y is the unit vector in the direction y, $\mathcal{P}$ is defined in (2.18) and $\mathcal{P}_2$ is an unknown pressure related to the almost incompressibility condition given by the second of the equations (1.20).

When δ goes to 0 the equations take a rather simpler form:

$$\begin{aligned}
&\hat{\nabla}\cdot\hat{u} = 0,\\
&\partial_t U + \hat{u}\cdot\hat{\nabla}U = \eta_0\hat{\Delta}U,\\
&\partial_t\hat{u} + \hat{u}\cdot\hat{\nabla}\hat{u} + \hat{\nabla}\mathcal{P}_2 - \frac{1}{C^2}\rho U^2 e_y = \eta_0\hat{\Delta}\hat{u},
\end{aligned} \tag{1.21}$$

$$\begin{aligned}
&\hat{\nabla}[r+\tau] = 0,\\
&\frac{3}{2}\partial_t\tau + \frac{5}{2}\hat{u}\cdot\hat{\nabla}\tau = \kappa_0\hat{\Delta}\tau.
\end{aligned} \tag{1.22}$$

The equations have to be completed with Cauchy initial data and the time independent boundary conditions

$$\hat{u}(\pm\pi, z) = 0, \quad U(\pm\pi, z) = U_\pm, \quad \tau(\pm\pi, z) = 0.$$

The first of the equations (1.22) is a Boussinesq condition ensuring the constancy of the pressure to the first order in δ, while the second is just the heat equation with a convective term. Note that the equations (1.21) are decoupled from the (1.22) and can be solved independently.

The equations (1.21) are the equations for the planar Couette system with an extra term in the equation for $\hat{u}$, representing the curvature ghost effect. The linear analysis for them [17, 19] shows the presence of a bifurcation controlled by the parameters C and $U_\pm$, with a stationary laminar solution losing its stability and bifurcating into two stable non laminar solutions.

This paper is devoted to the analysis of the 1-d stationary laminar solution to (1.5). The two dimensional case where bifurcation arises, will be presented in a forthcoming paper.

In Section 2 will be given a perturbative analysis in δ of the system (2.31) in order to control the difference between its solutions and those to the equations (2.32).

Due to the presence of the centrifugal force, the boundary layer expansion, as in [13, 1, 2], has to include that force. This requires the solution of a Milne problem with a force, which has been given in [10] in the presence of a potential force. The present force is different because it is not potential and depends on velocities. Therefore the arguments in [10] require several modifications which are presented in Section 3. Finally, in Section 4 we estimate the remainder. The key ingredient to do this is a generalized spectral inequality for a perturbed linearized Boltzmann operator, already used in [2] in the context of the Benard problem. The presence of the $\mathcal{O}(\delta)$ terms make the use of that inequality more involved and we need to assume δ sufficiently small (independent of ϵ). Due to the diffuse boundary conditions the analysis of the remainder is here done for the scaling $\delta^2 = \gamma\varepsilon$ for a suitably small γ independent of ε.

The main result of this paper is summarized in the following

Theorem 1.1. *Assume $\delta^2 = \gamma\varepsilon$. Then, for ε and γ small enough, there exists a positive isolated L_2-solution to the problem*

$$v_y\frac{\partial F}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2}\sigma(y)v_x\left(v_x\frac{\partial F}{\partial v_y} - v_y\frac{\partial F}{\partial v_x}\right) = \frac{1}{\varepsilon}Q(F,F), \tag{1.23}$$

with diffuse reflection boundary conditions

$$F(\mp\pi, v) = \mp\tilde{M}_\mp\int_{v_y<0} dv v_y F(\mp\pi, v), \quad \pm v_y > 0. \tag{1.24}$$

Moreover, for $q = 2$ and $q = \infty$,

$$\| \, [F - M(1+\delta v_x U)]M^{-1} \, \|_{q,2} \leq c\varepsilon, \tag{1.25}$$

where U is the unique solution to (2.32) and $\| \cdot \|_{q,2}$ is defined in (4.19).

Remark 1.1. The proof of the theorem shows that the rest term is of order ε^3 in L_∞.

2. **Expansions.** In this section we show how to compute the contributions F_n for $n = 1, \dots, N$ in (1.19). A modified Hilbert expansion is used to compute the bulk terms. Since they violate the boundary conditions, we introduce boundary layer corrections essentially supported in thin layers (of size of the order of ε) near the inner and outer cylinders. Therefore, F_n is written as follows:

$$F_n = B_n + b_n^+ + b_n^-, \tag{2.1}$$

where B_n is a smooth function of y, while $b_n^\pm$ are smooth exponentially fast decaying functions of the rescaled variables $Y^\pm = \varepsilon^{-1}(\pi \mp y)$, so that they are exponentially small away from $\pm\pi$.

2.1. **The bulk expansion.** In order to compute the expression of the B_n, we substitute (1.18) in (1.23). We ignore the terms $b_n^{\pm}$, because they are assumed exponentially small, and equate terms with the same power of ε up to the order N. We use the short notation

$$\mathcal{L}_\delta f = 2Q(M_\delta, f); \tag{2.2}$$

$$\mathcal{N}(f) = v_x \left(v_x \frac{\partial f}{\partial v_y} - v_y \frac{\partial f}{\partial v_x} \right). \tag{2.3}$$

Moreover, since in this subsection the parameter δ is kept fixed, we omit the index δ when there is no ambiguity. We get the following conditions:

$$\mathcal{L}B_1 = v_y \partial_y M, \tag{2.4}$$

$$\mathcal{L}B_2 = v_y \partial_y B_1 - Q(B_1, B_1), \tag{2.5}$$

$$\mathcal{L}B_3 = v_y \partial_y B_2 + \frac{1}{C^2\delta^2}\mathcal{N}(M) - 2Q(B_1, B_2), \tag{2.6}$$

$$\begin{aligned} \mathcal{L}B_n &= v_y \partial_y B_{n-1} + \frac{1}{C^2\delta^2}\sum_{h=0}^{n-3} \sigma^{(h)}\mathcal{N}(B_{n-3-h}) \\ &- \sum_{h,k\geq 1, h+k=n} Q(B_h, B_k), \quad n = 4, \dots, N. \end{aligned} \tag{2.7}$$

In (2.7) $B_0 \equiv M$ and $\sigma^{(h)}$ are the coefficients of the ε-power series expansion of $\sigma(y)$:

$$\sigma(y) = \sum_{h=0}^{\infty} \varepsilon^h \sigma^{(h)}(y).$$

The appropriate functional space to solve the above equations is the the Hilbert space $\mathcal{H}$ of the real measurable functions on the velocity space $\mathbb{R}^3$, equipped with the inner product

$$(f, g) = \int_{\mathbb{R}^3} dv M^{-1}(v) f(v) g(v). \tag{2.8}$$

The operator $\mathcal{L}$ is defined in a suitable dense submanifold $\mathcal{D}_\mathcal{L}$ of $\mathcal{H}$ and satisfies the following properties:

L1) $\mathcal{L}$ is symmetric and non positive: $(f, \mathcal{L}g) = (g, \mathcal{L}f)$; $(f, \mathcal{L}f) \leq 0$.

L2) $\mathcal{L}$ has a 5-dimensional null space spanned by the collision invariants:

$$\operatorname{Null}\mathcal{L} = \operatorname{span}\{\psi_0, \dots, \psi_4\}, \tag{2.9}$$

with $\psi_\beta = \chi_\beta M$, $\beta = 0, \dots, 4$ and

$$\chi_0 = 1; \quad \chi_1 = v_x; \quad \chi_2 = v_y; \quad \chi_3 = v_z; \quad \chi_4 = \frac{|v|^2}{2}. \tag{2.10}$$

The orthogonal projector on $\operatorname{Null}\mathcal{L}$ is denoted P, while $P^\perp = 1 - P$ denotes the projector on the orthogonal complement of $\operatorname{Null}\mathcal{L}$ in $\mathcal{H}$.

L3) The range of $\mathcal{L}$ is orthogonal to $\operatorname{Null}\mathcal{L}$: $(\psi_\alpha, \mathcal{L}f) = 0$ for any $\alpha = 0, \dots, 4$ and for any $f \in \mathcal{D}_\mathcal{L}$.

L4) The following decomposition holds:

$$\mathcal{L}f = -\nu f + \mathcal{K}f \tag{2.11}$$

where $\mathcal{K}$ is a compact operator and ν a smooth function such that

$$\nu_0(1 + |v|) \leq \nu(v) \leq \nu_1(1 + |v|) \text{ for all } v \in \mathbb{R}^3. \tag{2.12}$$

L5) If $g \in P^\perp \mathcal{H}$ then, by the Fredholm alternative theorem and L4), there is a unique solution in $P^\perp \mathcal{H}$ to the equation

$$\mathcal{L} f = g, \tag{2.13}$$

which, with a slight abuse of notation, we denote by $\mathcal{L}^{-1} g$:

$$f = \mathcal{L}^{-1} g. \tag{2.14}$$

Any solution in $\mathcal{H}$ of (2.13) can be written as

$$f = \mathcal{L}^{-1} g + \bar{f} \tag{2.15}$$

with $\bar{f} \in \mathrm{Null}\,\mathcal{L}$.

L6) Spectral inequality: there is a constant $c > 0$ such that

$$(f, \mathcal{L} f) \le -c(P^\perp f, \nu P^\perp f). \tag{2.16}$$

By L3) in order to solve (2.4) we need to impose the compatibility condition $P(\hat{v} \cdot \hat{\nabla} M) = 0$. It is immediate to check that this is true if and only if

$$\partial_y \mathcal{P} = 0, \tag{2.17}$$

where

$$\mathcal{P} = \rho T, \quad \rho = 1 + \delta r, \quad T = 1 + \delta \tau. \tag{2.18}$$

This is just the second of (2.31).

If this condition is satisfied, then

$$v_y \partial_y M = \delta(\tilde{\mathfrak{A}} \partial_y U + \tilde{\mathfrak{B}} \partial_y \tau), \tag{2.19}$$

where

$$\begin{aligned} \tilde{\mathfrak{B}} &= \bar{v}_x v_y M, \\ \tilde{\mathfrak{A}} &= \frac{\bar{v}^2 - 5T}{2T^2} v_y M \end{aligned} \tag{2.20}$$

are in $P^\perp \mathcal{H}$. We define $\mathfrak{A}$ and $\mathfrak{B}$ as the solutions in $P^\perp \mathcal{H}$ of the equations

$$\mathcal{L} \mathfrak{A} = \tilde{\mathfrak{A}}; \quad \mathcal{L} \mathfrak{B} = \tilde{\mathfrak{B}}. \tag{2.21}$$

Therefore, by L5)

$$\mathfrak{A} = \mathcal{L}^{-1} \tilde{\mathfrak{A}}, \quad \mathfrak{B} = \mathcal{L}^{-1} \tilde{\mathfrak{B}}. \tag{2.22}$$

Henceforth,

$$B_1 = \delta(\mathfrak{B} \partial_y U + \mathfrak{A} \partial_y \tau) + M \left(\frac{\rho_1}{\rho} + \frac{\bar{v} \cdot u}{T} + \frac{\bar{v}^2 - 3T}{2T^2} \tau_1 \right) := B_1^\perp + B_1^\parallel, \tag{2.23}$$

with ρ_1, τ_1 and u to be determined by the other conditions.

In order to solve (2.5), we need to impose the orthogonality of the right hand side of (2.6) to the null space of $\mathcal{L}$.

A standard computation shows that this is equivalent to the conditions

$$\begin{aligned} &\partial_y(\rho u_y) = 0 \\ &\rho\,(u_y \partial_y U) = \partial_y(\eta \partial_y U), \\ &\partial_y(T \rho_1 + \rho \tau_1) = 0, \\ &\frac{5}{2} \rho u_y \partial_y \tau = \partial_y(\kappa \partial_y \tau) + \delta \eta (\partial_y U)^2, \end{aligned} \tag{2.24}$$

with

$$\eta = -\int_{\mathbb{R}^3} dv \mathfrak{B}\tilde{\mathfrak{B}}, \quad \kappa = -\int_{\mathbb{R}^3} dv \mathfrak{A}\tilde{\mathfrak{A}}.$$

Then we can compute the part $B_2^\perp$ of B_2 as before:

$$B_2 = -\mathcal{L}^{-1}Q(B_1, B_1) + \mathcal{L}^{-1}P^\perp(\hat{v}\cdot\hat{\nabla}B_1) + B_2^\parallel, \tag{2.25}$$

with

$$B_2^\parallel = M\left(\frac{\rho_2}{\rho} + \frac{\bar{v}\cdot u_2}{T} + \frac{\bar{v}^2 - 3T}{2T^2}\tau_2\right), \tag{2.26}$$

and ρ_2, u_2 and τ_2 to be determined.

The same procedure is applied to solve the equation for B_3, (2.6). The compatibility condition is

$$P\Big(v_y\partial_y B_2 + \frac{1}{C^2\delta^2}\mathcal{N}(M)\Big) = 0. \tag{2.27}$$

This implies several conditions. We write explicitly only the following:

$$\rho u_y\partial_y u_y + \partial_y \mathcal{P}_2 - \frac{1}{C^2}\rho U^2 = \partial_y\left(\eta\partial_y u_y + \frac{\delta^2}{\mathcal{P}}\left[\sigma_1(\partial_y\tau)^2 + \sigma_2(\partial_y U)^2\right]\right). \tag{2.28}$$

Here σ_i are some suitable transport coefficients of higher order whose explicit expression is given for example in [17].

Note that the term in U^2 in equation (2.28) derives from the contribution $\int dv v_y\mathcal{N}(M) = \mathcal{O}(\delta^2)$. The result in (2.28) is independent of δ because of the scaling (1.15) and hence it persists in the limit $\delta \to 0$.

The equations written so far are just the system (2.31). They represent a system in the unknown functions U, u_y, τ and r, which does not include any of the extra funtions ρ_1, τ_1 etc, which also have to satisfy some extra conditions, for example the Boussinesq condition to the first order in ε.

We do not give the explicit conditions which follow in a rather standard way. It is clear that the above procedure can be continued to any specific order. In this paper it will be truncated at $N = 5$. Note that at each step the solution is given up to the choice of five arbitrary functions which are fixed in the subsequent steps. In particular the last term of the truncated expansion, B_N, is determined up to its hydrodynamic part which is arbitrary. We will take advantage of this arbitrariness when dealing with the equation for the remainder.

2.2. **The boundary layer expansion.** We need to include in our scheme a boundary layer expansion, because the bulk terms B_1 do not satisfy the diffuse reflection boundary condition. For example, it is immediate to check from equation (2.23) that B_1 cannot be proportional to the Maxwellian $\tilde{M}_\pm$ at the boundaries. Therefore we introduce the corrective terms $b_1^\pm$, with a fast dependence on y, so that they are sensibly different from 0 only close to the boundary. To achieve this, $b_1^\pm$ is assumed to be a smooth function of the variable $Y^\pm = \varepsilon^{-1}(\pi \mp y)$. We define $\bar{b}_1^\pm$ as the solution to the following equation:

$$v_y\frac{\partial\bar{b}_1^\pm}{\partial Y^\pm} + \frac{\varepsilon^3}{C^2\delta^2}\tilde{\sigma}(\mp(\varepsilon Y^\pm - \pi))\mathcal{N}(\bar{b}_1^\pm) = \mathcal{L}^\pm\bar{b}_1^\pm + \tilde{\mathcal{L}}_\vartheta^\pm\bar{b}_1^\pm, \tag{2.29}$$

with $\mathcal{L}^\pm g = 2Q(M_\pm, g)$, $M_\pm = M(1, 1, (\delta U_\pm, 0, 0); v)$. Here $\tilde{\sigma} = \sigma\varphi$, with φ a smooth cutoff function

$$\varphi(y) = \begin{cases} 1 & y \in [0, \zeta], \\ 0 & y > 2\zeta \end{cases}$$

for some $\zeta > 0$ and with uniformly bounded derivatives. The operator $\tilde{\mathcal{L}}^{\pm}_{\vartheta}$ is defined in the same way as $\mathcal{L}^{\pm}$, but we replace the hard spheres collision cross section $B(n, V)$ with $\vartheta B(n, V)^2$. The reason for introducing this unphysical operator is due to a technical difficulty which will be discussed in the next section. The parameter ϑ is chosen as $\vartheta = \frac{\varepsilon^3}{C^2\delta^2}$, and the contribution from $\tilde{\mathcal{L}}^{\pm}_{\vartheta}$, which should not be there, will be subtracted in the next order of the boundary layer expansion. We prescribe vanishing mass flux at the boundary:

$$m_1^{\pm} = \int_{\mathbb{R}^3} dv v_y \bar{b}_1^{\pm}(0, v) = 0.$$

This equation has to be solved with prescribed incoming data at $Y^{\pm} = 0$:

$$\bar{b}_1^{\pm}(0, v) = h_1^{\pm}(v), \qquad \text{for } v_y > 0.$$

The incoming boundary data are chosen in such a way to compensate the fact that $B_1^{\perp}$ is not proportional to a Maxwellian at the boundary: $h^{\pm}(v) = -B_1^{\perp}(\pm\pi)$. The solution to the Milne problem in general has a finite, but not vanishing limit at infinity, achieved exponentially fast. Let it be denoted by $\bar{b}^{\pm}_{1,\infty}$. It belongs to the null space of $\mathcal{L}^{\pm}$. The non vanishing of $\bar{b}^{\pm}_{1,\infty}$ is not good to our purposes because this contributes to the solution in the bulk. Therefore we define $b_1^{\pm} = \bar{b}_1^{\pm} - \bar{b}^{\pm}_{1,\infty}$. In this way we ensure the decay at infinity, but $b_1^{\pm}$ do not satisfy any more the equation (2.29) , because a term of the form $\frac{\varepsilon^3}{C^2\delta^2}\tilde{\sigma}(\mp(\varepsilon Y^{\pm} - \pi))\mathcal{N}(\bar{b}^{\pm}_{1,\infty})$ appears in the right hand side. We will compensate it with a term in the next order of the boundary layer expansion. We are not yet done, because the boundary value of $F_1^{\pm}$ would still be incorrect for the term $-\bar{b}^{\pm}_{1,\infty}$ which is not Maxwellian. However, we have not yet fixed the boundary values of $B_1^{\parallel}$ and we can use them to compensate it, since it is in the null space of $\mathcal{L}^{\pm}$. Finally, note that, on each boundary, the boundary value correction due to the other boundary is not zero, but exponentially small in ε^{-1}. This will be compensated in the remainder. In conclusion $f_1 = B_1 + b_1^+ + b_1^-$ satisfies the diffuse reflection boundary conditions up to terms $\Psi_1^{\pm}$ exponentially small in ε^{-1}.

The equation (2.29) is a special case of the Milne problem we discuss in the next section. We note however that in the standard Milne problem the second term in the left hand side of (2.29) is absent. When in the equation there is a force term, as in the present case and in [13, 1, 2], although very small, the lack of regularity in the velocity of the solution to the Milne problem for $v_y = 0$ (the derivative $\partial_{v_y} b_1^{\pm}$ does not exist for $v_y = 0$ at the boundary), does not allow us to include them in higher order terms of the expansion. Indeed we need to keep it in (2.29) which will be solved in a suitably weak sense, because we cannot afford to have any v_y derivative of $b_1^{\pm}$ present in the expansion. This problem was already present in the case of the Benard problem where the force derives from a potential and the solution is given in [10].

The corrections to B_n, for $n > 1$ will be $b_n^{\pm}$ solving a similar equation:

$$v_y \frac{\partial \bar{b}_n^{\pm}}{\partial Y^{\pm}} + \frac{\varepsilon^3}{C^2\delta^2}\bar{\sigma}(\mp(\varepsilon Y^{\pm} - \pi))\mathcal{N}(\bar{b}_n^{\pm}) = \mathcal{L}^{\pm}\bar{b}_n^{\pm} + \tilde{\mathcal{L}}^{\pm}_{\vartheta}\bar{b}_n^{\pm} + S_n^{\pm}, \tag{2.30}$$

with prescribed incoming data at $Y^{\pm} = 0$:

$$\bar{b}_n^{\pm}(0, v) = h_n^{\pm}(v), \qquad \text{for } v_y > 0,$$

vanishing mass flux at the boundary:

$$m_n^{\pm} = \int_{\mathbb{R}^3} dv v_y \bar{b}_n^{\pm}(0, v) = 0$$

and then define $b_n^{\pm} = \bar{b}_n^{\pm} - \bar{b}_{n,\infty}^{\pm}$. The incoming boundary data are chosen in such a way that $f_n = B_n + b_n^+ + b_n^-$ satisfies the diffuse reflection boundary conditions up to terms exponentially small in ε^{-1}, $\Psi_n^{\pm}$. The source term $S_n^{\pm}$ has the following form, for $n > 1$:

$$S_n^{\pm} = -\tilde{\mathcal{L}}_{\vartheta}^{\pm} \bar{b}_{n-1}^{\pm} + \sum_{h,k \geq 1, h+k=n} \Big[2Q(B_h, b_k^{\pm}) + Q(b_h^{\pm}, b_k^{\pm}) + Q(b_h^{\pm}, b_k^{\mp}) \\ - \frac{\varepsilon^2}{C^2 \delta^2} \tilde{\sigma}^c (\mp(\varepsilon Y^{\pm} - \pi)) \mathcal{N}(b_{n-1}^{\pm}) - \frac{\varepsilon^2}{C^2 \delta^2} \tilde{\sigma}(\mp(\varepsilon Y^{\pm} - \pi)) \mathcal{N}(\bar{b}_{n-1,\infty}^{\pm}) \Big]$$

with $\tilde{\sigma}^c = \sigma(1 - \varphi)$ and $b_0^{\pm} = 0$ and the property $\int dv S_n^{\pm}(y, v) = 0$. The existence of the solutions to (2.29), (2.30) with the prescribed conditions follows from Theorem 3.1, given in next section, via a procedure which is the same used in [12, 13, 1, 2]. We do not repeat it here and refer to those papers for details.

2.3. **Hydrodynamical expansion.** In this section we compare the solution of the stationary 1-d equations for $\delta > 0$ with the solution of the limiting equations for $\delta = 0$. The former are

$$\begin{aligned}
&\partial_y(\rho u_y) = 0, \\
&\partial_y[(1 + \delta\rho)(1 + \delta\tau)] = 0, \\
&\rho u_y \partial_y U = \partial_y(\eta \partial_y U), \\
&\frac{5}{2}\rho u_y \partial_y \tau = \partial_y(\kappa \partial_y \tau) + \delta(\eta \partial_y U)^2, \\
&\rho u_y \partial_y u_y + \partial_y \mathcal{P}_2 - \frac{1}{C^2}\rho U^2 = \eta_0 \partial_y^2 u_y \\
&+\partial_y \left(\eta_\delta \partial_y u_y + \frac{\delta^2}{\mathcal{P}} \left[\sigma_1 (\partial_y \tau)^2 + \sigma_2 (\partial_y U)^2 \right] \right),
\end{aligned} \tag{2.31}$$

with $\rho = 1 + \delta r$ and the boundary conditions

$$u_y(\pm\pi) = 0; \quad U_x(\pm\pi) = U_{\pm}, \quad \tau(\pm\pi) = r(\pm\pi) = 0.$$

In the limit of vanishing δ the velocity u_y and τ are identically zero and

$$\eta_0 \partial_y^2 U = 0, \tag{2.32}$$

$$\partial_y \mathcal{P}_2 - \frac{1}{C^2}\rho U^2 = 0$$

whose solution is the laminar field $\bar{U} = U_- + \beta(y + \pi)$, with $\beta = (2\pi)^{-1}(U_+ - U_-)$.

The first equation in (2.31) implies, by using the boundary conditions for u_y, $\rho u_y = 0$. Since $\rho = 1 + \delta r$ for δ small is strictly larger than zero, it has to be $u_y = 0$. The equations reduce to

$$\begin{aligned}
&\partial_y(\eta \partial_y U) = 0, \\
&\partial_y(\kappa \partial_y \tau) + \delta(\eta \partial_y U)^2 = 0, \\
&\partial_y \mathcal{P}_2 - \frac{1}{C^2}\rho U^2 = \frac{\delta}{\mathcal{P}} \partial_y(\sigma_1 (\partial_y \tau)^2).
\end{aligned} \tag{2.33}$$

The first two equations decouple from the third and can be solved to find U and τ. Then the last one gives $\mathcal{P}_2$.

We define $\tilde{U} = U - \bar{U}$. We have

$$\begin{aligned} &\partial_y(\eta\partial_y\tilde{U}) + \beta\partial_y\eta = 0,\\ &\partial_y(\kappa\partial_y\tau) + \delta\eta(\beta + \partial_y\tilde{U})^2 = 0,\\ &\tilde{U}(\pm\pi) = 0; \quad \tau(\pm\pi) = 0. \end{aligned} \tag{2.34}$$

The functions η and κ are smooth functions of the temperature. The solutions are constructed by an iterative procedure. We therefore assume that $\|\tau\|_\infty < 1$. In consequence $\|\eta\|_\infty$ and $\|\kappa\|_\infty$ are uniformly bounded for $\delta < 1$. Moreover, for δ sufficiently small, $\|\eta\|_\infty > \frac{\eta_0}{2}$ and $\|\kappa\|_\infty > \frac{\kappa_0}{2}$ with η_0 and κ_0 the values of η and κ for $\tau = 0$. Finally, note that $\partial_y\eta = \delta\eta'\partial_y\tau$ with $\eta' = \frac{\partial\eta}{\partial T}$ uniformly bounded. By multiplying the first of (2.34) by $\tilde{U}$ and the second by τ and integrating in y, after an integration by parts we get:

$$\begin{aligned} &\frac{\eta_0}{2}\|\partial_y\tilde{U}\|^2 \le c\delta\|\tilde{U}\|\|\partial_y\tau\| \le c\delta\|\partial\tilde{U}\|\|\partial_y\tau\|,\\ &\frac{\kappa_0}{2}\|\partial_y\tau\|^2 \le c\delta\|\tau\|_\infty(\beta^2 + \|\partial\tilde{U}\|^2) \le c\delta\|\partial\tau\|(\beta^2 + \|\partial\tilde{U}\|^2). \end{aligned}$$

Therefore

$$\begin{aligned} &\|\partial_y\tilde{U}\| \le c\delta\|\partial_y\tau\|,\\ &\|\partial_y\tau\| \le c\delta(\beta^2 + \|\partial_y\tilde{U}\|^2). \end{aligned}$$

The above inequalities imply that $\|\partial_y\tilde{U}\|$ and $\|\partial_y\tau\|$ are $\mathcal{O}(\delta)$. In particular $\|\partial_y\tau\|_\infty < 1$ for δ sufficiently small. We omit the proof of the convergence of the approximating sequence, which follows along the same lines.

We conclude with the following

Theorem 2.1. *If δ is sufficiently small, the equations (2.31) have a unique $C^\infty(-\pi,\pi)$ stationary solution which differs from the laminar solution (namely $U = \bar{U}(y)$, $\tau = 0$) for $\mathcal{O}(\delta)$.*

Call $(\rho^\delta, U^\delta, u^\delta_y, \tau^\delta)$ the solution of (2.31) (remember $u^\delta_y = 0$) and $(1, U, 0, 1)$ the solution of the equations for $\delta = 0$ with U the unique solution of (2.32).
Let $M_\delta = M(\rho^\delta, (\delta U^\delta, 0), \tau^\delta)$ and $M^0_\delta = M(1, (\delta U, 0), 1)$. Then

Corollary 2.2. *For $q = 2, \infty$*

$$\| M^{-1}[M_\delta - M^0_\delta] \|_{q,2} \le c\delta^2, \quad \| M^{-1}[M^0_\delta - M] \|_{q,2} \le c\delta.$$

2.4. **Estimates for the expansion.** The properties of F_n's constructed in this section are summarized in the following theorem:

Theorem 2.3. *The functions F_n, $n = 1, \dots, 5$ and $\Psi_{n,\varepsilon}$ can be determined so as to satisfy the boundary conditions*

$$\begin{aligned} F_n(\mp\pi, v) \qquad &= M_\mp(v)\int_{w_y \lessgtr 0} |w_y|[F_n(\mp\pi, w) - \Psi_{n,\varepsilon}(\mp\pi, w)]dw\\ &+\Psi_{n,\varepsilon}(\mp\pi, v), \;\; v_z \gtrless 0, \end{aligned}$$

and the normalization condition $\int_{\mathbb{R}^3\times[-\pi,\pi]} dvdyF_n = 0$, so that the asymptotic expansion in ε for the stationary problem (1.23), truncated to the order 5 is given *by*

$$\Phi = \sum_{n=1}^{5} \varepsilon^n F_n(y,v).$$

The functions F_n's satisfy the conditions

$$\| M^{-1}F_n \|_{2,2} < \infty, \quad \| M^{-1}F_n \|_{\infty,2} < \infty\ , \quad n = 1,\dots,5.$$

The functions $\Psi_{n,\varepsilon}$ are such that $\|\Psi_{n,\varepsilon}\|_{q,2,\sim}$, $q = 2,\infty$, are exponentially small as $\varepsilon \to 0$ and $\int_{\mathbb{R}^3} dv v_y \Psi_{n,\varepsilon} = 0$.

Proof. It follows as in [1, 2], using the Theorem 3.1 in next section. □

3. **Milne problem.** In this section we deal with the Milne problem

$$\begin{aligned}
&v_y \frac{\partial g}{\partial Y} + G\omega(Y)Ng = Lg + S,\\
&g(0,v) = h(v), \quad v_y > 0,\\
&\int dv v_y M(v) g(0,v) = 0,\\
&\int dv S(y,v) M = 0
\end{aligned} \tag{3.1}$$

for S and h prescribed. Here M is the Maxwellian with $T = 1$, $\rho = 1$ and $u = (\mathfrak{U}, 0, 0)$;

$$Lf = 2M^{-1}[\mathcal{L}(Mf) + \mathcal{L}_\vartheta(Mf)], \quad Nf = M^{-1}\mathcal{N}(Mf).$$

and $\omega(Y)$ a compactly supported smooth function. Moreover, we assume that there is $c > 0$ such that

$$\int_{v_y>0} dv v_y M(v) h^2(v) < c. \tag{3.2}$$

Note the explicit expression of Nf:

$$Nf = v_x^2 \frac{\partial f}{\partial v_y} - v_x v_y \frac{\partial f}{\partial v_x} - \mathfrak{U} v_x v_y f. \tag{3.3}$$

The results will be applied with $G = \frac{\varepsilon^3}{\delta^2 C^2}$, $\vartheta > G\mathfrak{U}$, $\omega = \tilde{\sigma}$ and $\mathfrak{U} = \delta U_\pm$. The procedure is the same used in [10]: we construct the solution in a slab of size ℓ with reflecting boundary condition $g(\ell, Rv) = g(\ell, v)$, $Rv = (v_x, -v_y, v_z)$ and obtain estimates uniform in ℓ, then we take the limit $\ell \to \infty$. As in [10], the main point is to discuss the case $S = 0$. The assumptions on S will be given in Theorem 3.1 below. We only point out the differences with [10]. We write $g = q + w$ with $q \in$ Null L and $(q, w) = 0$. We set $\bar{\chi}_0 = 1$, $\bar{\chi}_1 = v_x - \mathfrak{U}$, $\bar{\chi}_2 = v_y$, $\bar{\chi}_3 = v_z$, $\bar{\chi}_4 = \frac{1}{2}[(v_x - \mathfrak{U})^2 + v_y^2 + v_z^2]$, and $q = \sum_{\alpha=0}^4 b_\alpha(Y)\bar{\chi}_\alpha$. Moreover, $(\,\cdot\,,\,\cdot\,)$ denotes the inner product on $L_2(\mathbb{R}^3, Mdv)$.

Note that $b_2 = 0$. Indeed, by multiplying (3.1) by M and integrating in dv, we get

$$\frac{\partial}{\partial Y}(v_y, g) + G\omega(v_y, g) = 0,$$

because

$$(1, Ng) = \int dv g M v_y.$$

Therefore, with $\Omega(Y) = \int_0^Y dY' \omega(Y')$, we have

$$b_2(Y) = (v_y, g)(Y) = \exp\{-G[\Omega(\ell) - \Omega(Y)]\}(v_y, g)(\ell) = 0$$

because $(v_y, g)(\ell)$ vanishes by the reflecting boundary conditions at $Y = \ell$.

As a consequence, $q = \sum_{\alpha\neq 2} b_\alpha(Y)\bar\chi_\alpha$. Moreover, $(v_y\bar\chi_\alpha, \bar\chi_\beta) = 0$ for $\alpha, \beta \neq 2$. Therefore $(v_y q, q) = 0$.

Set $I_\alpha = (v_y\bar\chi_\alpha, g) = (v_y\bar\chi_\alpha, w)$ for $\alpha \neq 2$. The functions I_α satisfy the following equations:

$$\frac{\partial}{\partial Y} I_\alpha = \omega \sum_{\beta\neq 2} \mathcal{S}_{\alpha,\beta} I_\beta,$$

with

$$\mathcal{S}_{\alpha,\beta} = \delta_{\alpha,\beta} + (\delta_{\alpha,1} - \mathfrak{U}\delta_{\alpha,4})\delta_{1,\beta},$$

for $\alpha, \beta \neq 2$. Indeed,

$$(\bar\chi_\alpha, N\bar\chi_\beta) = \int dv M\bar\chi_\alpha\Big(\mathfrak{U} v_x v_y(\delta_{\beta,4} - 1) - v_x v_y \delta_{\beta,1}\Big),$$

which is odd in v_y for $\alpha \neq 2$. On the other hand,

$$(\bar\chi_\alpha, Nw) = \int dv\Big[v_y M\bar\chi_\alpha w + M v_y v_x w(\delta_{\alpha,1} - \mathfrak{U}\delta_{\alpha,4})\Big].$$

Therefore, with $I = (I_\alpha)_{\alpha\neq 2}$, we have

$$I(Y) = I(\ell)\exp\{(\Omega(\ell) - \Omega(Y))\mathcal{S}\},$$

which implies

$$I(Y) = 0$$

by the reflection boundary condition at $Y = \ell$.

We take the inner product of the first equation in (3.1) with g. By using (3.3),

$$(g, N(g)) = \frac{1}{2}(v_y g, g) - \frac{1}{2}\mathfrak{U}(v_x v_y g, g),$$

and so we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial Y}(v_y g, g) + \frac{1}{2}G\omega(v_y g, g) &= (w, Lw) + \frac{1}{2}G\mathfrak{U}\omega(v_x v_y g, g)\\ &= (w, Lw) + \frac{1}{2}G\mathfrak{U}\omega(v_x v_y w, w) + G\mathfrak{U}\omega(v_x v_y w, q). \quad (3.4)\end{aligned}$$

because $(v_x v_y q, q) = 0$. The last but one term in the above Green identity is handled by adding in the cross section in the linearized Boltzmann operator the unphysical term $\vartheta B(\,\cdot\,,\,\cdot\,)^2$. Indeed, in this case the inequality (2.16) holds with ν replaced by $\nu + \vartheta\tilde\nu$, with ν satisfying the inequalities (2.12) and $\tilde\nu$ such that

$$\tilde\nu_0(1 + |v|)^2 \le \tilde\nu(v) \le \tilde\nu_1(1 + |v|)^2, \quad v \in \mathbb{R}^3 \tag{3.5}$$

for suitable constants $\tilde\nu_0$ and $\tilde\nu_1$. This allows to control the term $\frac{G\mathfrak{U}\omega}{2}(v_x v_y w, w)$ as long as we have $\vartheta > \frac{G\mathfrak{U}}{2}$.

So we have only to worry about the term $\omega G\mathfrak{U}(v_x v_y q, w)$ for which we use the bound

$$\omega G\mathfrak{U}|(v_x v_y q, w)| \le \frac{1}{2}\omega G\mathfrak{U}\|w\|^2 + \frac{c}{2}\omega G\mathfrak{U}\|q\|^2.$$

We set $\mathcal{A} = (v_y g, g)$. Note that $\mathcal{A}(0) < c$ by (3.2). We need upper and lower bounds on $\mathcal{A}(0)$. We can write

$$\mathcal{A}(0) = \mathcal{A}(\ell)\exp\{\frac{G}{2}(\Omega(\ell)\} + \int_0^\ell dY \exp\{\frac{G}{2}(\Omega(\ell) - \Omega(Y)\}$$
$$\Big[-(w, (L + \frac{1}{2}\mathfrak{U}G\omega v_x v_y)w)(Y) - G\mathfrak{U}\omega(Y)(v_x v_y q, w)\Big] \quad (3.6)$$

By the reflecting boundary conditions, $\mathcal{A}(\ell) = 0$. Moreover, $(w, (L+\frac{1}{2}G\mathfrak{U}\omega v_x v_y)w) \geq -(w, \nu w)$ for $\vartheta > \frac{1}{2}G\mathfrak{U}$. We now need to estimate the last term. We use the following bound proved later:

$$\|q(Y)\| \leq \sqrt{c + |\mathcal{A}(0)|} + c\|\sqrt{\nu}w\|(Y) + \int_0^Y dY'\|\sqrt{\nu}w\|(Y'). \quad (3.7)$$

We use the bound

$$\int_0^\ell \omega dY \left[\int_0^Y ||w||(Y')dY'\right]^2 \leq \int_0^\ell \omega(Y)YdY \int_0^\ell ||w||^2(Y')dY'$$
$$\leq c\int_0^Y ||w||^2(Y')dY'. \quad (3.8)$$

Then we plug (3.7) in (3.6) and use the spectral inequality, to get the following bound for $|\mathcal{A}(0)|$, using the fact that ω is compactly supported,

$$|\mathcal{A}(0)| \leq G(c + c|\mathcal{A}(0)|),$$

which implies the boud on $|\mathcal{A}(0)|$ for G sufficiently small. Using this one can conclude that

$$\int_0^\ell dY\|w(Y)\|^2 < c \quad (3.9)$$

uniformly in ℓ.

We need to prove the bound (3.7). Let $\beta_\alpha = (v_y^2\bar{\chi}_\alpha q)$ for $\alpha \neq 2$. Since $\beta_\alpha = \sum_{\gamma\neq 2} A_{\alpha,\gamma}b_\gamma$, with $A_{\alpha,\gamma} = (v_y^2\bar{\chi}_\alpha, \bar{\chi}_\gamma)$ a positive non singular matrix, to estimate β_α is equivalent to estimate $\|q\|$. The equation for β_α is obtained by taking the inner product of the first equation in (3.1) with $v_y\bar{\chi}_\alpha$. The result is

$$\frac{\partial\beta_\alpha}{\partial Y} = G\omega\sum_{\gamma\neq 2}\mathcal{B}_{\alpha,\gamma}\beta_\gamma + \mathcal{D}_\alpha + (v_y\bar{\chi}_\alpha, Lw),$$

with

$$\mathcal{D}_\alpha = -\frac{\partial}{\partial Y}(v_y^2\bar{\chi}_\alpha, w) - G\omega(v_y\bar{\chi}_\alpha, Nw)$$

and

$$\mathcal{B}_{\alpha,\gamma} = G\omega\Big[\delta_{\alpha,\gamma}(1 + \mathfrak{U}(\delta_{\alpha,1} - U\delta_{\alpha,4}) - (v_x^2\bar{\chi}_\alpha, \bar{\chi}_\gamma) + (\delta_{\alpha,1} - \mathfrak{U}\delta_{\alpha,4})(\bar{v}_x^2, v_y^2)A^{-1}_{\alpha,\gamma}\delta_{1,\gamma}\Big].$$

An integration by parts shows that $|(v_y\bar{\chi}_\alpha, Nw)| \leq c\|w\|$. The rest of the argument is as in [10]. The only difference is in the estimate of $\beta_\alpha(0)$. We have

$$|(v_y^2\bar{\chi}_\alpha, g)(0)| \leq (|v_y|g, g)^{1/2}(|v_y|^3, |\bar{\chi}_\alpha|).$$

$$(|v_y|g, g)(0) = \int_{v_y>0} v_y h^2 - \int_{v_y<0} v_y g^2 = 2\int_{v_y>0} v_y h^2 - \mathcal{A}(0).$$

By using (3.2) we then get (3.7).

To get estimates uniform in ℓ also for q we take the scalar product of the first equation in (3.1) equation by $L^{-1}(\bar{\chi}_\alpha v_y)$. The term on the right hand side is $(L^{-1}(\bar{\chi}_\alpha v_y), Lw)$, which is zero by the orthogonality property. We get then an equation for $\Theta_\alpha = (v_y L^{-1}(\bar{\chi}_\alpha v_y), q)$ whose solution is

$$\Theta(Y) = e^{-\int_0^Y dsG\omega(s)(\mathcal{G}\mathcal{Q}^{-1})}(L^{-1}(v_y\bar{\chi}), v_y g)(0) - (L^{-1}(v_y\bar{\chi}), v_y w)(Y)$$
$$+ \int_0^Y dt e^{-\int_t^Y dsG\omega(s)(\mathcal{G}\mathcal{Q}^{-1})} D(t)$$

where $\mathcal{G}$ and $\mathcal{Q}$ are suitable matrix, with $\Theta_\alpha = \mathcal{Q}_{\alpha\gamma} b_\gamma$, $\mathcal{Q}$ invertible, and

$$D_\alpha(Y) = -G\omega(Y)\Big[(L^{-1}(v_y\bar{\chi}_\alpha), Nw) - \frac{\partial}{\partial Y}(L^{-1}(v_y\bar{\chi}_\alpha), v_y w)\Big].$$

By Schwartz inequality, boundedness of L^{-1} and the fact that ω has compact support, the last integral is finite. Moreover, $\| w \|$ vanishes at infinity and the first term on the right hand side is finite. This implies that there exists a finite limit at infinity, Θ^∞, of Θ and there is Y_0 such that for $Y > Y_0$ we have

$$|\Theta(Y) - \Theta^\infty|^2 \leq c||w||^2(Y).$$

By the argument in [10] we get also for $Y > Y_0$

$$|b(Y) - b^\infty|^2 \leq c||w||^2(Y).$$

The other arguments in [10] can be adapted in a similar way. Using above estimates, the exponential decay of $\|w\|$ and $|b_\alpha - b_\alpha^\infty|$ is established. The properties of the derivatives are also obtained with the method presented there, with a minor modification due to the special structure of the force in this case. Indeed, the v_x and v_y derivatives of g satisfy in this case a coupled system of equations. Therefore, they are both controlled only away from the boundary.

To state the final theorem, we define the norms

$$\| f\|_{q,2,\theta} = \left(\int_{\mathbb{R}^3} dvM(v)\left(\int_\theta^\infty dY|f(Y,v)|^q\right)^{\frac{2}{q}}\right)^{\frac{1}{2}}, \tag{3.10}$$

for $\theta \geq 0$.

Theorem 3.1.
1) Suppose that for some $\beta > 0$

$$\| e^{\beta Y} S \|_{2,2,0} < \infty, \quad \| e^{\beta Y} S \|_{\infty,2,0} < \infty .$$

Then there is a unique solution $g \in L_2(\mathbb{R}^3, L_\infty(\mathbb{R}^+)) \cap L_2(\mathbb{R}^+ \times \mathbb{R}^3)$ to the Milne problem (3.1). Moreover there exist constants c and c' such that g verifies the conditions:

$$\int_{\mathbb{R}^3} Mgdv = 0, \quad g_\infty \in \mathrm{Null}\,\tilde{L},$$

$$\| e^{\beta' Y}(g(Y,v) - g_\infty(v)) \|_{2,2} < c, \quad \| e^{\beta' Y}(g(Y,v) - g_\infty(v)) \|_{\infty,2} < c$$

for any $\beta' < c'$.
2) Suppose that for $\ell \geq 1$, $\theta > 0$ and $i = 1,\dots,3$

$$\int_{v_y>0} M\Big|\frac{\partial^\ell h}{\partial v_i^\ell}\Big| < \infty, \quad \Big\|e^{\beta Y}\frac{\partial^\ell S}{\partial v_i^\ell}\Big\|_{2,2,\theta} < \infty, \quad \Big\|e^{\beta Y}\frac{\partial^\ell S}{\partial v_i^\ell}\Big\|_{\infty,2,\theta} < \infty.$$

Then there are constants c' and c_ℓ such that

$$\Big\| e^{\beta' Y} \frac{\partial^\ell g}{\partial v_i^\ell} \Big\|_{2,2,\theta} < c_\ell, \quad \Big\| e^{\beta' Y} \frac{\partial^\ell g}{\partial v_i^\ell} \Big\|_{\infty,2,\theta} < c_\ell.$$

for any $\beta' < c'$.

4. **The remainder.**

4.1. **Equation for the remainder.** It is immediate to check that the remainder $\mathcal{R}$ in (1.18) has to satisfy the following equation:

$$v_y \frac{\partial \mathcal{R}}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2} \sigma(y) v_x \left(v_x \frac{\partial \mathcal{R}}{\partial v_y} - v_y \frac{\partial \mathcal{R}}{\partial v_x} \right) = \frac{1}{\varepsilon} \Big[\mathcal{L}_\delta \mathcal{R} + 2Q(\Phi, \mathcal{R}) \Big] + \varepsilon Q(\mathcal{R}, \mathcal{R}) + A, \tag{4.1}$$

with the inhomogeneous term A given by

$$\begin{aligned} A = & \sum_{h,k \geq 1, h+k > N} \varepsilon^{h+k-2} Q(F_h, F_k) - \varepsilon^{N-1} v_y \partial_y B_N - \\ & - \frac{\varepsilon^2}{C^2 \delta^2} \sum_{h=N-2}^{\infty} \sum_{n=0}^{N} \varepsilon^{h+n} \sigma^{(h)} \mathcal{N}(B_n) - \frac{\varepsilon^2}{C^2 \delta^2} \sum_{h=0}^{N-3} \varepsilon^h \sigma^{(h)} \sum_{n=N-2}^{N} \varepsilon^n \mathcal{N}(B_{n-h}) \\ & - \varepsilon^N \sum_{\pm} \Big[\tilde{\mathcal{L}}_\vartheta^\pm b_N^\pm + \frac{\varepsilon^3}{C^2 \delta^2} \tilde{\sigma}^c (\mp(\varepsilon Y^\pm - \pi)) \mathcal{N}(\bar{b}_{N,\infty}^\pm) \Big], \quad \vartheta = \frac{\varepsilon^3}{C^2 \delta^2} \end{aligned} \tag{4.2}$$

We remind that $M_\delta = M(1 + \delta r, 1 + \delta \tau, (\delta U, 0, 0); v)$, where $(r, \tau, (U, 0, 0))$ is the solution of (2.31). We keep now the dependence on δ and denote by M the standard Maxwellian. We write $\mathcal{R} = MR$ and denote by L the operator

$$Lf = M^{-1} \mathcal{L} M f. \tag{4.3}$$

We use the Hilbert space $\mathbb{H}$ of the measurable functions on the velocity space $\mathbb{R}^3$ with inner product

$$(f, g) = \int_{\mathbb{R}^3} dv M(v) f(v) g(v). \tag{4.4}$$

Note that this involves M, while the one considered in Section 2, (2.8) involved M_δ^{-1}. The operator L has the same properties already mentioned for $\mathcal{L}_\delta$ and we do not repeat them here. We just note that the null space of L is given by the span of the χ_β's defined in (2.10). We still denote the projector on it by P and on its orthogonal complement by $P^\perp = 1 - P$.

Moreover, we write

$$M^{-1} \mathcal{L}_\delta f = Lf + M^{-1} [\mathcal{L}_\delta - \mathcal{L}] M f. \tag{4.5}$$

We also use the notation

$$J(f, g) = M^{-1} Q(Mf, Mg). \tag{4.6}$$

The following operator will play a major role:

$$L_J f = Lf + \Gamma f, \quad \Gamma f = 2J(W, Pf), \quad W = \frac{M_\delta - M}{M} + \frac{\Phi}{M}. \tag{4.7}$$

By Theorem 2.3 and Corollary 2.2, $W = \mathcal{O}(\delta + \varepsilon)$ and hence the operator Γ is $\mathcal{O}(\delta + \epsilon)$.

In [2] it is proved the following

Theorem 4.1. *The operator L_J has the following properties:*

1. *The null space of L_J is the span of the functions*
$$\psi^J_\beta = \chi_\beta - L^{-1}\Gamma\chi_\beta, \quad \beta = 0,\dots,4; \tag{4.8}$$
P_J denotes the projection on it.
2. *The null space of its adjoint L^*_J is the same as L.*
3. *If δ and ε are sufficiently small, the following spectral inequality holds:*
$$(f, L_J f) \le -c((1-P_J)f, \nu(1-P_J)f) \tag{4.9}$$
for a suitable positive constant c.

Using these notation, we write the equation (4.1) for the remainder as follows:
$$v_y\frac{\partial R}{\partial y} + \frac{\varepsilon^2}{c^2}\sigma(y)v_x\left(v_x\frac{\partial R}{\partial v_y} - v_y\frac{\partial R}{\partial v_x}\right) = \frac{1}{\varepsilon}L_J R + \frac{1}{\varepsilon}H_1 R + \varepsilon J(R,R) + a, \tag{4.10}$$
with $a = M^{-1}A$ and
$$H_1 f = 2J(W,(1-P)f). \tag{4.11}$$
Note that, thanks to the arbitrariness left in the determination of the F_N, we can assume that
$$Pa = 0. \tag{4.12}$$
The equation for the remainder has to be solved with the boundary conditions:
$$\begin{aligned} &R(-\pi,v) = \alpha_-(R)M^{-1}\tilde{M}_- - \frac{1}{\varepsilon}\Psi(-\pi,v), \quad v_y > 0\\ &R(\pi,v) = \alpha_+(R)M^{-1}\tilde{M}_+ - \frac{1}{\varepsilon}\Psi(\pi,v), \quad v_y < 0, \end{aligned} \tag{4.13}$$
where
$$M\Psi(\mp\pi,z,v,t) = \sum_{n=1}^{n}\varepsilon^n\Psi_{n,\varepsilon}(\mp\pi,z,v,t) \tag{4.14}$$
which are exponentially small in ε^{-1} because of Theorem 2.3, and
$$\begin{aligned} &\alpha_-(R) = -\int_{v_y<0} dv v_y M[R(-\pi,v) + \frac{1}{\varepsilon}\Psi(-\pi,v)],\\ &\alpha_+(R) = \int_{v_y>0} dv v_y M[R(\pi,v) + \frac{1}{\varepsilon}\Psi(\pi,v)]. \end{aligned} \tag{4.15}$$

4.2. **Estimates for the remainder.** We will proceed with the construction of the solution by iteration, based on estimates for a linearized problem where the non linear term $J(R,R)$ is computed at the previous step of the iteration. The generic term of the iteration will then satisfy a linear equation of the type
$$v_y\frac{\partial R}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2}\sigma(y)v_x(v_x\frac{\partial R}{\partial v_y} - v_y\frac{\partial R}{\partial v_x}) = \frac{1}{\varepsilon}[L_J R + H_1(R)] + g. \tag{4.16}$$
At the step n of the iteration the term g will be replaced by $a + \varepsilon J(R^{n-1}, R^{n-1})$, therefore we assume
$$Pg = 0. \tag{4.17}$$

The boundary conditions are

$$R(\mp\pi, z, v) = \frac{M_\mp}{M} \int_{\mp w_y > 0} (R(\mp\pi, w) + \frac{1}{\varepsilon}\Psi(\mp\pi, w))|w_y| M dw - \frac{1}{\varepsilon}\Psi(\mp\pi, w)), \quad \pm v_y > 0. \quad (4.18)$$

In order to simplify the argument we also assume that the velocity of the inner cylinder vanishes, so that M_- is the standard Maxwellian, up to the normalization. The argument can be easily adapted to the general case.

We first consider the linear problem with g given and will get L_2 bounds. This is done in two steps: first, we get a control of the non-hydrodynamic part in terms of the hydrodynamic one. Second, we get an estimate of the hydrodynamic part in terms of the non-hydrodynamic one and then a bound for R in terms of the known term g.

The norms used below are defined as follows:

$$\| f\|_{q,2} = \left(\int_{\mathbb{R}^3} dv M(v) \left(\int_{[-\pi,\pi]} dy |f(y,v)|^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}. \quad (4.19)$$

Defining the ingoing velocity spaces $\mathbb{R}^3_\pm$ at $y = \mp\pi$ as the sets $v = (v_x, v_y, v_z)$ such that $v_z \gtrless 0$

$$\| f \|_{q,2,\sim} = \sup_{\pm} \left(\int_{\mathbb{R}^3_\pm} dv |v_y| M(v) \left(|f(\mp\pi, v)|^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}. \quad (4.20)$$

We use the traces

$$\gamma^\pm f = \begin{cases} f|_{y=-\pi}, & \text{if } v_y \in \mathbb{R}^3_\pm, \\ f|_{y=\pi}, & \text{if } v_y \in \mathbb{R}^3_\mp. \end{cases} \quad (4.21)$$

Note that the norm $\| \cdot \|_{2,2,\sim}$ is defined only for incoming velocities. In the sequel, with an abuse of notation we will denote by $\| \gamma^- f \|_{2,2,\sim}$ the $\| \cdot \|_{2,2,\sim}$-norm of $S\gamma^- f$, where S is the reflection of the y component of the velocity.

We now establish the lemmas which allow to bound the solution to (4.16) in these norms.

- **Step** 1

Lemma 4.2. *Assume $\delta^2 = \varepsilon\gamma$. Then, for γ small enough and for any $\eta > 0$, the solution of (4.16) satisfies*

$$\frac{1}{\varepsilon} \| \nu^{\frac{1}{2}} (I - P_J) R \|^2_{2,2} \leq c \Big[(\varepsilon\gamma + \varepsilon\eta) \| P_J R \|^2_{2,2} + \frac{1}{\eta\varepsilon} \| P_J g \|^2_{2,2} + \frac{1}{\eta\varepsilon^3} \| \Psi \|^2_{2,2,\sim} + \varepsilon \| (I - P_J) g \|_{2,2} \Big]^2, \quad (4.22)$$

$$\| \gamma^- R \|^2_{2,2\sim} \leq c \big[\eta \| P_J R \|^2_{2,2} + \frac{1}{\eta\varepsilon^2} \| (I - P_J) R \|^2_{2,2} + \varepsilon^2 \| (I - P_J) g \|^2_{2,2} + \| P_J g \|^2_{2,2} + \frac{1}{\eta\varepsilon^3} \| \Psi \|^2_{2,2,\sim} \big]. \quad (4.23)$$

Proof. Define

$$k(y) = \exp\left(\int_{-\pi}^{y} \frac{\varepsilon^2}{\delta^2 C^2}\sigma(q)dq\right). \tag{4.24}$$

Multiply (4.16) by $2RMk(y)$ and integrate over v. We get

$$\frac{\partial}{\partial y}(v_y R, kR) = 2k[(R, L_J R) + (R, H_1(R)) + (R, g)].$$

Introduce $\tilde R = R\sqrt{k(y)}$,$\tilde g = g\sqrt{k(y)}$ and $\tilde\Psi = \Psi\sqrt{k(y)}$.

$$\frac{\partial}{\partial y}(v_y \tilde R, \tilde R) = 2\big[(\tilde R, L_J \tilde R) + (\tilde R, H_1(\tilde R)) + (\tilde R, \tilde g)\big].$$

We bound all the terms in the r.h.s. in the following way: We write

$$L_J\tilde R = \frac{1}{2}L\tilde R + \frac{1}{2}L\tilde R + \Gamma\tilde R = \frac{1}{2}L\tilde R + \hat L_J\tilde R.$$

We notice that $\hat L_J$ also satisfies the same spectral inequality as L_J but with a different constant and for smaller δ and ε. We use this spectral inequality and the one for L to get

$$(\tilde R, L_J\tilde R) \le -c \parallel \nu^{\frac12}(I-P)\tilde R \parallel_2^2 -c \parallel \nu^{\frac12}(I-P_J)\tilde R \parallel_2^2 .$$

We use the fact that H_1 depends only on $(I-P)\tilde R$, $J(\cdot,\cdot)$ is orthogonal to the null space of L and that $H_1(\tilde R)$ is of order $\delta+\varepsilon$ to get the bound

$$(\tilde R, H_1(R)) \le c(\delta+\varepsilon) \parallel \nu^{\frac12}(I-P)\tilde R \parallel_2^2,$$

so that

$$(\tilde R, L_J\tilde R) + (\tilde R, H_1(\tilde R)) \le -c \parallel \nu^{\frac12}(I-P_J)\tilde R \parallel_2^2 .$$

Moreover,

$$(\tilde R, \tilde g) = ((I-P_J)\tilde R, (I-P_J)\tilde g) + (P_J\tilde R, P_J\tilde g) \le c\Big[\frac{1}{\varepsilon\eta} \parallel (I-P_J)\tilde R \parallel_2^2$$

$$+\varepsilon\eta \parallel (I-P_J)\tilde g \parallel_2^2 + \eta\varepsilon \parallel P_J\tilde R \parallel_2^2 + \frac{1}{\varepsilon\eta} \parallel P_J\tilde g \parallel_2^2 \Big],$$

with $\eta > 0$ an arbitrary number.

Next, integrate over $[-\pi,\pi]\times\mathbb{R}^3$. Putting all the estimates together,

$$\begin{aligned} -\mathcal{B} + c_1\frac{1}{\varepsilon} \parallel \nu^{\frac12}(I-P_J)\tilde R \parallel_{2,2}^2 \le c\Big[\varepsilon \parallel (I-P_J)\tilde g \parallel_{2,2}^2 \\ + \frac{1}{\varepsilon\eta} \parallel P_J\tilde g \parallel_{2,2}^2 + \varepsilon\eta \parallel P_J\tilde R \parallel_{2,2}^2 \Big], \end{aligned} \tag{4.25}$$

where $\mathcal{B} := (v_y, \tilde R^2(-\pi, v)) - (v_y, \tilde R^2(\pi, v))$.

Following the argument in [13] one can show that the first term in $\mathcal{B}$ is nonpositive and the second can be bounded as

$$\begin{aligned} -\int_{\mathbb{R}^3} Mv_y\tilde R^2(\pi,v)dv \le c(\delta^2+\varepsilon\eta)\int_{v_y>0} M|v_y|\tilde R^2(\pi,v)dv \\ + \frac{1}{\varepsilon^3\eta}\int_{v_y>0} M|v_y|\tilde\Psi^2(\pi,v)dv. \end{aligned} \tag{4.26}$$

Replacing in (4.25) we get

$$\frac{1}{\varepsilon} \parallel \nu^{\frac{1}{2}}(I-P_J)\tilde{R} \parallel_{2,2}^2 \leq c\Big[\varepsilon \parallel (I-P_J)\tilde{g} \parallel_{2,2}^2 + \frac{1}{\varepsilon\eta} \parallel P_J\tilde{g} \parallel_{2,2}^2 \tag{4.27}$$
$$+\eta\varepsilon \parallel P_J\tilde{R} \parallel_{2,2}^2 + (\delta^2+\varepsilon\eta)\int_{v_y>0} M|v_y|\tilde{R}^2(\pi,v)dv\Big]$$
$$+\frac{1}{\varepsilon^3\eta}\int_{v_y>0} M|v_y|\tilde{\Psi}^2(\pi,v)dv.$$

To estimate the outgoing integral of $\tilde{R}$ in the right hand side, multiply (4.16) by $M\tilde{R}\sqrt{k}$, integrate in velocity over the region $v_y \geq q$, then over space using a smooth cut-off function $\chi(y)$ which is zero close to $y=-\pi$, and equal to one close to π, and finally over q for $q_0 \leq q \leq 0$ for $|q_0|$ small enough. Since we do not integrate over all $v \in \mathbb{R}^3$ we cannot use the spectral inequality to control the terms involving L_J and H_1, but we use only the boundedness of L_J. We get

$$\int_{q_0}^0 dq \int_{v_y\geq q} dvMv_y\tilde{R}^2(\pi,v) \leq c|q_0|\Big[(\eta+\delta^2+\varepsilon^2) \parallel P_J\tilde{R} \parallel_{2,2}^2$$
$$+\frac{1}{\eta\varepsilon^2} \parallel (I-P_J)\tilde{R} \parallel_{2,2}^2 + \varepsilon^2 \parallel (I-P_J)\tilde{g} \parallel_{2,2}^2 + \frac{1}{\eta} \parallel P_J\tilde{g} \parallel_{2,2}^2 \Big] \tag{4.28}$$
$$+\int_{q_0}^0 dq \int_{v_y\geq q} dvdyM\chi'(y)v_y\tilde{R}^2(y,v).$$

The term on the l.h.s. equals

$$|q_0| \parallel \gamma^-\tilde{R}(\pi) \parallel_{2,2\sim}^2 + \int_{q_0}^0 dq \int_{q\leq v_y<0} dvv_yM\tilde{R}^2(\pi,v),$$

where

$$\gamma^-\tilde{R}(\pi) = \tilde{R}(\pi,v), \quad v_y>0.$$

We move the second term in the previous expression to the r.h.s. of (4.28) and bound it, by replacing $\tilde{R}$ by the ingoing boundary data, as

$$\Bigg|\int_{q_0}^0 dq \int_{0\leq v_y<q} \frac{1}{M}v_y dv \Bigg[M^+\int_{w_y\geq 0} dww_yM\big(\tilde{R}(\pi,w)+\frac{1}{\varepsilon}\Psi(\pi,w)\big)$$
$$-M\frac{1}{\varepsilon}\Psi(\pi,v)\Bigg]^2\Bigg| \leq c(q_0)\big[\parallel \gamma^-\tilde{R}(\pi) \parallel_{2,2\sim}^2 + \frac{1}{\varepsilon^2} \parallel \tilde{\Psi} \parallel_{2,2,\sim}^2 \big], \tag{4.29}$$

with $c(q_0)=o(|q_0|)$. We replace this estimate in (4.28) and divide both sides by $|q_0|$

$$\parallel \gamma^-\tilde{R}(\pi) \parallel_{2,2\sim}^2 \leq c\big[(\eta+\delta^2+\varepsilon^2) \parallel P_J\tilde{R} \parallel_{2,2}^2 + \frac{1}{\eta\varepsilon^2} \parallel (I-P_J)\tilde{R} \parallel_{2,2}^2$$
$$+\varepsilon^2 \parallel (I-P_J)\tilde{g} \parallel_{2,2}^2 + \frac{1}{\eta} \parallel P_J\tilde{g} \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel \tilde{\Psi} \parallel_{2,2,\sim}^2 \big]. \tag{4.30}$$

We have estimated the term

$$-\int_0^{q_0} dq \int_{v_y\geq q} dvdyM\chi'(y)v_y\tilde{R}^2(y,v)$$

as $\eta_1|q_0| \parallel \tilde{R} \parallel_{2,2}^2$, with η_1 small, by using a suitable cut-off function χ with positive derivative $\chi'<1$ and the bound $-v_y \leq -q$ in the integral.

Finally, we use the above bound for $\| \gamma^- \tilde{R} \|^2_{2,2\sim}$ in (4.27) to get,

$$c_1 \frac{1}{\varepsilon} \| \nu^{\frac{1}{2}} (I - P_J) \tilde{R} \|^2_{2,2} \leq [\varepsilon\gamma + \varepsilon\eta] \| P_J \tilde{R} \|^2_{2,2} + \frac{1}{\varepsilon\eta} \| P_J \tilde{g} \|^2_{2,2} + \frac{1}{\varepsilon^3 \eta} \| \Psi \|^2_{2,2,\sim} + \varepsilon \| (I - P_J) \tilde{g} \|^2_{2,2} . \quad (4.31)$$

To deal with the two terms

$$\delta^2 \eta \| P_J \tilde{R} \|^2_{2,2}, \quad \frac{\delta^2}{\eta \varepsilon^2} \| (I - P_J) \tilde{R} \|^2_{2,2} .$$

we had to choose $\delta^2 = \varepsilon\gamma$, with γ a small number. With this choice the second term gets a factor $\frac{\gamma}{\eta\varepsilon}$ which can be absorbed on the l.h.s for γ small and the first one gets a factor $\varepsilon\gamma\eta$. We remark that this last term comes from the estimate of the outgoing flux and is due to the fact that the Maxwellians on the two boundaries differ by a factor δ.

Now it is easy to bound also $\gamma^- \tilde{R}(-\pi)$ by similar arguments and get

$$\| \gamma^- \tilde{R}(\pi) \|^2_{2,2\sim} \leq c[\eta \| P_J \tilde{R} \|^2_{2,2} + \frac{1}{\eta\varepsilon^2} \| (I - P_J) \tilde{R} \|^2_{2,2} + \varepsilon^2 \| (I - P_J) \tilde{g} \|^2_{2,2} + \frac{1}{\eta} \| P_J \tilde{g} \|^2_{2,2} + \frac{1}{\varepsilon^2} \| \tilde{\Psi} \|^2_{2,2,\sim}]. \quad (4.32)$$

Finally, we notice that the norms involving $\tilde{R}, \tilde{g}$ are equivalent to the ones for R, g since $k(y)$ defined in (4.24) is a function uniformly bounded in ε. □

- **Step** 2

We will provide an a priori estimate for the hydrodynamic part of R. The equation for the remainder is

$$v_y \frac{\partial R}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2} \sigma(y) v_x (v_x \frac{\partial R}{\partial v_y} - v_y \frac{\partial R}{\partial v_x}) = \frac{1}{\varepsilon} [L_J R + H_1(R)] + g \quad (4.33)$$

with the boundary conditions

$$R(\mp\pi, v) = -\frac{1}{\varepsilon} \Psi(\mp\pi, w) + \frac{M_\mp}{M} \int_{\mp w_y > 0} (R(\mp\pi, w) + \frac{1}{\varepsilon} \Psi(\mp\pi, w)) |w_y| M dw, \ \pm v_y > 0 \quad (4.34)$$

where now in the definition of L_J the function Φ is determined by Theorem 2.3.

We simplify the problem by changing the boundary conditions in π from diffusive to given ingoing data. The following lemma states the equivalence between the two problems.

Lemma 4.3. *Equation (4.33) with the new boundary conditions*

$$\begin{aligned} R(-\pi, z, v) &= \frac{M_-}{M} \int_{-w_y > 0} (R(-\pi, w) + \frac{1}{\varepsilon} \Psi(-\pi, z, w)) |w_y| M dw, \\ &\quad - \frac{1}{\varepsilon} \Psi(-\pi, w), \quad v_y > 0 \qquad (4.35) \\ R(\pi, z, v) &= \frac{M_+}{M} \int_{w_y > 0} \frac{1}{\varepsilon} \Psi(\pi, w) |w_y| M dw - \frac{1}{\varepsilon} \Psi(\pi, v), \quad v_y < 0 \end{aligned}$$

has the same solution as problem (4.33)-(4.34)

Proof. We start by noticing that existence and uniqueness for the new problem are classical. The main point is that the new boundary condition implies

$$\int_{v_y>0} R(\pi,v)v_y M dv = 0.$$

In fact, by multiplication of (4.33) by $k(y)M$ and integration over $v \in \mathbb{R}^3$, since $\int gMdv = 0$, we get

$$\frac{\partial}{\partial y}\left(k(y)\int v_y MRdv\right) = 0, \tag{4.36}$$

which implies

$$k(\pi)\int v_y MR(\pi,v)dv = k(-\pi)\int v_y MR(-\pi,v)dv.$$

Hence, since $\int_{\mathbb{R}^3} v_y MR(-\pi,v)dv = 0$, we have also $\int v_y MR(\pi,v)dv = 0$. We write the integral in the left hand side by using the boundary condition (4.35)

$$\begin{aligned} 0 \quad &= \int_{\mathbb{R}^3} v_y MR(\pi,v)dv = \int_{v_y<0} v_y M_+ dv \int_{w_y>0} \frac{1}{\varepsilon}\Psi(\pi,w))w_y M dw \qquad (4.37)\\ &+ \int_{v_y>0} v_y MR(\pi,v)dv - \int_{v_y<0} v_y \frac{1}{\varepsilon}\Psi(\pi,v) = \int_{v_y>0} v_y MR(\pi,v)dv \end{aligned}$$

because $\int_{v_y<0} v_y M_+ dv = -1$ and $\int_{\mathbb{R}^3} v_y \Psi(\pi,v) = 0$.
Hence, $\int_{v_y>0} R(\pi,v)v_y M dv = 0$ and this implies that the unique solution R of the new problem also satisfies the old boundary conditions (4.34). $\square$

Remark 4.1. Note that the condition $\int_{\mathbb{R}^3} R(\pi,v)v_y M dv = 0$ is crucial to make this argument work. However, while the old boundary conditions are constructed in such a way that this is true, the new boundary conditions do not ensure that it is automatically satisfied. It is the conservation law (4.36) which ensures the vanishing of the outgoing flux also with the new boundary conditions.

Lemma 4.4. *If $\int Mgdv = 0$, then the solution of (4.33) satisfies*

$$\| P_J R \|_{2,2}^2 \le c\Big[\frac{1}{\varepsilon^2} \| (I-P_J)R \|_{2,2}^2 + \| g \|_{2,2}^2 + \frac{1}{\varepsilon^2} \| \Psi \|_{2,2,\sim}^2 \Big]. \tag{4.38}$$

Proof. Let $\hat{R} = \mathcal{F}_y R$, be the Fourier-transform in y of R and $\xi_y \in \mathbb{Z}$ the conjugate variable to y. $\hat{R}$ satisfies the equation

$$iv_y\xi_y\hat{R} + \frac{\varepsilon^2}{\delta^2 C^2}\mathcal{F}_y\left\{\sigma(y)v_x(v_x\frac{\partial R}{\partial v_y} - v_y\frac{\partial R}{\partial v_x})\right\} = \varepsilon^{-1}\widehat{L_J R} + \widehat{H_1(R)} - v_y r(-1)^{\xi_y} + \hat{g} \tag{4.39}$$

with $r(v)$ now denoting the difference between ingoing and outgoing boundary values,

$$r(v) = R(\pi,v) - R(-\pi,v). \tag{4.40}$$

With the notations $< R >$ for the 0-Fourier term, and $\bar{R} := R- < R >$, we shall first give an estimate for $\bar{R}$. We use the notation

$$\begin{aligned} \tilde{Z} &= \varepsilon^{-1}\widehat{L_J R} + \widehat{H}_1(R) - \frac{\varepsilon^2}{\delta^2 C^2}\mathcal{F}_y\left\{\sigma(y)v_x(v_x\frac{\partial R}{\partial v_y} - v_y\frac{\partial R}{\partial v_x})\right\} + \hat{g} - (-1)^{\xi_y}v_y r,\\ Z &= \varepsilon^{-1}\widehat{L_J R} + \widehat{H}_1(R) + \hat{g} - (-1)^{\xi_y}v_y r,\\ Z' &= \varepsilon^{-1}\widehat{L_J R} + \widehat{H}_1(R) + \hat{g},\ \hat{U} = (i\xi_y v_y)^{-1}. \end{aligned}$$

Let $\mathfrak{h}$ be the indicatrix function of the set $\{v \in \mathbb{R}^3 \ |v_y| < \alpha\}$, for some positive α to be chosen later.

Let $\zeta_s(v) = (1+ \mid v \mid)^s$. For $\xi_y \neq 0$

$$\begin{aligned} \parallel P(\mathfrak{h}\hat{R}(\xi_y, \cdot) \parallel \qquad & \leq c \sum_{j=0}^{4} \left| \int_{\mathbb{R}^3} dv \mathfrak{h}(v) \hat{R}(\xi_y, v) M(v) \chi_j(v) \right| \\ & \leq c \parallel \zeta_{-s} \mathfrak{h} \hat{R} \parallel \sum_{j=0}^{4} \parallel \mathfrak{h} \zeta_s \chi_j \parallel \leq c\sqrt{\alpha} \parallel \zeta_{-s} \mathfrak{h} \hat{R} \parallel . \end{aligned}$$

We use this estimate with the following choice of α, $\alpha = \parallel \zeta_{-s}\hat{R} \parallel_2^{-1} \parallel \zeta_{-s} Z' \parallel$. We also introduce an indicatrix function $\mathfrak{h}_1$ with $\alpha = \delta_1$. We fix δ_1 so that $c\delta_1 << 1$. Then we find from the above estimate that the P-part of the right-hand side, $\parallel P(\mathfrak{h}_1\hat{R}) \parallel$, can be absorbed by $\parallel P(\mathfrak{h}_1\hat{R}) \parallel$ in the left-hand side. The estimates hold in the same way when $\mathfrak{h}_1$ is suitably smoothed around $\sqrt{\delta_1}|\xi|$. For the remaining $\mathfrak{h}^c\mathfrak{h}_1^c\hat{R} = (1-\mathfrak{h})(1-\mathfrak{h})\hat{R}$, we shall use that $\hat{R} = -\hat{U}\tilde{Z}$. Then

$$\begin{aligned} & \parallel P(\mathfrak{h}_1^c\mathfrak{h}^c\hat{R}(\xi_y, \cdot)) \parallel^2 \\ & \leq c \parallel \zeta_{s+2}\mathfrak{h}_1^c\mathfrak{h}^c\hat{U} \parallel^2 \parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel P(\mathfrak{h}_1^c\mathfrak{h}^c v_y r) \parallel^2}{\delta_1|\xi_y|^2} + \frac{\varepsilon^2}{\delta^2 C^2}|\Theta| \\ & \leq \frac{c}{\mid \xi_y \mid^2 \mid \alpha \mid} \parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel P(\mathfrak{h}_1^c\mathfrak{h}^c v_y r) \parallel^2}{\delta_1|\xi_y|^2} + \frac{\varepsilon^2}{\delta^2 C^2}|\Theta|, \end{aligned}$$

with

$$\Theta = \sum_{j=0}^{4} \int \chi_j \mathfrak{h}_1^c\mathfrak{h}^c \hat{U} \mathfrak{F}_y \left\{ \sigma(y) v_x (v_x \frac{\partial R}{\partial v_y} - v_y \frac{\partial R}{\partial v_x}) \right\} M dv \left(\int \chi_j \mathfrak{h}_1^c\mathfrak{h}^c (\hat{R} - \hat{U}Z) M dv \right)^* .$$

We replace α by $\parallel \zeta_{-s}\hat{R} \parallel^{-1} \parallel \zeta_{-s}Z' \parallel$ in the denominator. That gives

$$\begin{aligned} \parallel P\hat{R}(\xi_y, \cdot) \parallel^2 \qquad & \leq c(\parallel \zeta_{-s}\hat{R} \parallel \parallel \zeta_{-s}Z' \parallel + \frac{\parallel P(\mathfrak{h}_1^c\mathfrak{h}^c v_y r) \parallel^2}{\delta_1|\xi_y|^2} \\ & + \delta_1 \parallel \zeta_{-s}(I-P)\hat{R} \parallel^2) + \frac{\varepsilon^2}{\delta^2 C^2}|\Theta|. \end{aligned}$$

Hence,

$$\begin{aligned} & \parallel P\hat{R}(\xi_y, \cdot) \parallel^2 \leq c\Big((\parallel P\hat{R} \parallel + \parallel \zeta_{-s}(I-P)\hat{R} \parallel)_2 \parallel \zeta_{-s}Z' \parallel \\ & + \frac{\parallel P(\mathfrak{h}_1^c\mathfrak{h}^c v_y r) \parallel^2}{\delta_1|\xi_y|^2} + \delta_1 \parallel \zeta_{-s}(I-P)\hat{R} \parallel^2 \Big) + \frac{\varepsilon^2}{\delta^2 C^2}|\Theta|. \end{aligned}$$

Consequently,

$$\begin{aligned} & \parallel P\hat{R}(\xi_y, \cdot) \parallel^2 \leq c\Big(\parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel P(\mathfrak{h}_1^c\mathfrak{h}^c v_y r) \parallel^2}{\delta_1|\xi_y|^2} + \parallel \zeta_{-s}(I-P)\hat{R} \parallel \parallel \zeta_{-s}Z' \parallel \\ & + \parallel \zeta_{-s}(I-P)\hat{R} \parallel^2 \Big) + \frac{\varepsilon^2}{\delta^2 C^2}|\Theta|. \end{aligned}$$

We next discuss the term $\frac{\varepsilon^2}{\delta^2 C^2}|\Theta|$. The first integral can be bounded by an integral of a product of M, $1+|\xi_y|$, a polynomial in v, $\mid \hat{R} \mid$ and $\hat{U}$ or $\hat{U}^2$. Since, by our choice of δ, we have $\frac{\varepsilon^2}{\delta^2 C^2} = \frac{\varepsilon}{\gamma^2 C^2}$, this term is bounded by $\varepsilon c \parallel \hat{R} \parallel_2$. And so,

$$\parallel P\hat{R}(\xi_y, \cdot) \parallel^2 \leq c\Big(\parallel \zeta_{-s} Z' \parallel^2 + \frac{\parallel P(\mathfrak{h}_1^c \mathfrak{h}^c v_y r) \parallel^2}{\delta_1 |\xi_y|^2} + \parallel (I-P)\hat{R} \parallel^2 \Big).$$

Therefore for $\xi_y \neq 0$,

$$\int |P\hat{R}|^2(\xi_y, v) M dv \leq c\Big(\frac{1}{\varepsilon^2} \parallel \zeta_{-s}(v)\widehat{L_J R}(\xi_y, \cdot) \parallel_2^2 + \parallel (I-P)\hat{R}(\xi_y, \cdot) \parallel^2$$
$$+ \frac{\parallel P(\mathfrak{h}_1^c \mathfrak{h}^c v_y r) \parallel^2}{\delta_1 |\xi_y|^2} + \parallel \nu^{-\frac{1}{2}} \hat{g}(\xi_y, \cdot) \parallel^2 \Big). \tag{4.41}$$

The error from evaluating P instead of P_J is of order δ. We remind that the zero Fourier mode of $\overline{P\hat{R}}$ is zero by definition. Hence, taking ε small enough and summing over all $0 \neq \xi_y \in \mathbb{Z}$, implies by Parseval and the spectral estimate on L_J, that

$$\int (\overline{P_J R})^2(y, v) M dv dy \leq c\Big(\frac{1}{\varepsilon^2} \int \nu((I-P_J)R)^2(y, v) M dv dy$$
$$+ \frac{\parallel P_J(\mathfrak{h}_1^c \mathfrak{h}^c v_y r) \parallel^2}{\delta_1} + \int \nu^{-1} g^2(y, v) M dv dy + \delta^2 \parallel R \parallel^2 \Big). \tag{4.42}$$

The r-term can be expressed from (4.39) at $\xi_y = 0$

$$v_y r(v) = \frac{1}{\varepsilon}\widehat{L_J R}(0, v) - \frac{\varepsilon^2}{\delta^2 C^2}\mathcal{F}_y \Big\{ \sigma(y) v_x (v_x \frac{\partial R}{\partial v_y} - v_y \frac{\partial R}{\partial v_x}) \Big\} \Big|_{\xi_y = 0} \tag{4.43}$$
$$+ \hat{g}(0, v) + \widehat{H_1(R)}(0, v).$$

Inserting this into (4.42) results in

$$\int (\overline{P_J R})^2(y, v) M dv dy \qquad \leq c\Big(\frac{1}{\varepsilon^2} \int \nu((I-P_J)R)^2(y, v) M dv dy$$
$$+ \int \nu^{-1} g^2(y, v) M dv dy + (\delta^2 + \varepsilon^2) \parallel R \parallel_{2,2}^2 \Big).$$

We are left with the Fourier component $P_J \hat{R}(\xi_y)$ for $\xi_y = 0$. We estimate separately the $(I-P)$-component and the P-component of $P_J \hat{R}(0, \cdot)$. For $(I-P)P_J \hat{R}(0, \cdot)$ we obtain

$$\parallel (I-P)P_J \hat{R}(0, \cdot) \parallel \leq c(\varepsilon + \delta)(\parallel (I-P_J)R \parallel_{2,2} + \parallel P_J R \parallel_{2,2}).$$

For $PP_J \hat{R}$ it is enough to consider $P\hat{R}$, since they differ only by terms of order $(\varepsilon + \delta)$. We have $P\hat{R}(0, v) = \sum_{\ell=0}^4 \chi_\ell(v) \int dv M \psi_\ell \int dy R(y, v)$. We discuss each moment separately.

Given two functions $h(v)$ and $f(\,\cdot\,, v)$ we use the notation $f_h(\,\cdot\,) := \int dv h(v) f(\cdot, v)$. In particular, for $h = \chi_j$, $j = 0, \ldots, 4$, we also use the notation f_j.

- v_y-moment:

Multiply (4.16) by $k(y)M$ and integrate over velocity. Now integrate over $[-\pi, y]$ and $v \in \mathbb{R}^3$. Since $\int v_y M(v) R(\pm\pi, v) dv = 0$, we have from (4.16) $\hat{R}_{v_y}(0) = \int v_y M(v) R(y, v) dv dy = 0$.

- v_x-moment:

We estimate the moment $\hat{R}_{v_x v_y^2}(0)$ and then use that

$$\hat{R}_{v_x} = \hat{R}_{v_x v_y^2} - \hat{R}^{\perp}_{v_x v_y^2},$$

with $\hat{R}^{\perp} = (1-P)\hat{R}$.
Indeed, $\int MPRv_x v_y^2 dv = \int MRv_x dv$, because $\int Mv_x^2 v_y^2 dv = 1$. Multiply equation (4.33) by $Mv_x v_y$, integrate over $[y,\pi]$ and $v \in \mathbb{R}^3$ and then integrate over $y \in [-\pi,\pi]$. We get

$$\begin{aligned} & -\int_{-\pi}^{\pi} dy \int dv v_x v_y^2 MR(y,v)dv + 2\pi \int dv v_x v_y^2 MR(\pi,v)dv \\ & \leq c\left[\frac{\varepsilon}{\gamma C^2} \parallel R \parallel_{2,2} + \frac{1}{\varepsilon} \parallel (I-P_J)R \parallel_{2,2} + \parallel g \parallel_{2,2}\right]. \end{aligned} \tag{4.44}$$

In the boundary term the integral over $v_y < 0$ is easy because the boundary condition in π is given in terms of Ψ. To control the outgoing ($v_y > 0$) part, we multiply again equation (4.33) by $2\pi M v_x v_y$, integrate this time over $[-\pi,\pi]$ and $v \in \mathbb{R}^3, v_y > 0$. We get

$$\begin{aligned} & 2\pi \int_{v_y>0} dv v_x v_y^2 MR(\pi,v)dv - 2\pi \int_{v_y>0} dv v_x v_y^2 MR(-\pi,v)dv \\ & \leq c\left[\frac{\varepsilon}{\gamma C^2} \parallel R \parallel_{2,2} + \frac{1}{\varepsilon} \parallel (I-P_J)R \parallel_{2,2} + \parallel g \parallel_{2,2}\right]. \end{aligned} \tag{4.45}$$

To control the second term in l.h.s. of (4.45), we use the boundary condition in $-\pi$ and notice that the integral is zero. Finally, we replace the bound of the outgoing part given by equation (4.45) in (4.44). The result is

$$\begin{aligned} |\hat{R}_{v_x v_y^2}(0)|^2 \quad & = \left|\int_{-\pi}^{\pi} dy \int v_x v_y^2 MR(y,v)dv\right|^2 \leq c[\frac{\varepsilon^2}{\gamma^2 C^4} \parallel R \parallel_{2,2}^2 \\ & + \frac{1}{\varepsilon^2} \parallel (I-P_J)R \parallel_{2,2}^2 + \parallel g \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel \Psi \parallel_{2\sim}^2] \end{aligned} \tag{4.46}$$

and the same estimate holds for $|\hat{R}_{v_x}(0)|^2$.

- v_z-moment:

we get the same estimate for the v_z-moment and the proof is analogous, the only difference being that we start by multiplying (4.33) by $Mv_y v_z$.

- χ_4-moment:

We recall $\chi_4 = \frac{|v|^2-3}{\sqrt{6}}$ and we denote $\hat{R}_4(0) = \int_{-\pi}^{\pi} \psi_4 MR dv dy$. We notice that

$$\hat{R}_{v_y^2 \bar{A}}(0) = \frac{1}{\sqrt{6}} \hat{R}_4(0) \int v_y^2 v^2 \bar{A} M dv + \hat{R}^{\perp}_{v_y^2 \bar{A}}(0), \tag{4.47}$$

where $\bar{A}$ is the non-hydrodynamic solution to

$$L(v_y \bar{A}) = v_y(v^2 - 5). \tag{4.48}$$

To control $\hat{R}_{v_y^2 \bar{A}}(0)$ we multiply (4.33) by $Mv_z^2 \bar{A}$ and proceed as before: first, we consider the integral $\int_{-\pi}^{\pi} dy \int_y^{\pi} dy' \int_{\mathbb{R}^3} dv$, then we study $2\pi \int_{-\pi}^{\pi} dy \int_{v_y>0} dv$ and take the difference. Now, the analogous of the second term in (4.45) is

$$2\pi \int_{v_y>0} dv v_y^2 \bar{A} MR(-\pi,v)dv.$$

We use the boundary condition in $-\pi$ and observe that $\int v_y^2 \bar{A} M dv = 0$ by orthogonality and since the integral is even in v_y, also $\int_{v_y>0} v_y^2 \bar{A} M dv = 0$.

- χ_0-moment:

Since $\displaystyle\int dv v_y^2 R M dv = R_0 + \frac{2R_4}{\sqrt{6}} + \int dv v_y^2 R^{\perp} M dv$, and we already have estimated $\hat{R}_4(0)$, it is enough to estimate the moment $\hat{R}_{v_y^2}(0)$. To this end, multiply (4.33) by $v_y M$ and integrate over $[y,\pi]\times\mathbb{R}^3$. Since v_y is in the null of L, we do not get contribution from the L and H_1 terms on the right hand side. Moreover, by integration by parts, there is no contribution depending on PR in the force term. We have

$$
\begin{aligned}
&-\int dv v_y^2 M R(y,v) dv + 2\pi \int_{v_y<0} dv v_y^2 M R(\pi,v) dv + 2\pi \int_{v_y>0} dv v_y^2 M R(\pi,v) dv \\
&\le c\Big(\frac{\varepsilon}{\gamma C^2} \parallel (I-P)R \parallel_{2,2} + \parallel g \parallel_{2,2}\Big).
\end{aligned}
$$

The second term depends on the ingoing flow in π, which is given in terms of Ψ. To control the third term, we will as before cancel it by considering a new equation. Multiply (4.33) by $2\pi v_y M$ and integrate over $[-\pi,\pi]\times\{v\in\mathbb{R}^3 : v_y>0\}$.

$$
\begin{aligned}
&2\pi \int_{v_y>0} dv v_y^2 M R(\pi,v) dv - 2\pi \int_{v_y>0} dv v_y^2 M R(-\pi,v) dv \\
&\le c\left(\frac{\varepsilon}{\gamma C^2} \parallel R \parallel_{2,2} + \parallel g \parallel_{2,2} + \frac{1}{\varepsilon} \parallel (I-P_J)R \parallel_{2,2}\right).
\end{aligned}
$$

We get

$$
\begin{aligned}
- \quad &\int dv v_y^2 M R(y,v) dv + 2\pi \int_{v_y>0} v_y^2 M R(-\pi,v) dv \\
&\le c\left(\frac{\varepsilon}{\gamma C^2} \parallel R \parallel_2 + \parallel g \parallel_2 + \frac{1}{\varepsilon} \parallel (I-P_J)R \parallel_2 + \frac{1}{\varepsilon} \parallel \Psi \parallel_{2\sim}\right).
\end{aligned}
$$

To estimate the second term, we cannot use symmetry arguments as before. We instead use Schwartz inequality

$$
\begin{aligned}
\int_{v_y>0} v_y^2 M |R|(-\pi,v) dv \le \left[\int_{v_y>0} v_y^3 M\right]^{1/2} \left[\int_{v_y>0} v_y M R^2(-\pi,v) dv\right]^{1/2} \\
\le c \parallel \gamma^- R \parallel_2 .
\end{aligned}
$$

We now can use (4.32) to estimate $\parallel \gamma^- R \parallel_2$ and get

$$
|\hat{R}_{v_y^2}|^2 \le c[\eta \parallel P_J R \parallel_{2,2}^2 + \frac{1}{\eta\varepsilon^2} \parallel (I-P_J)R \parallel_{2,2}^2 + \parallel g \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel \Psi \parallel_{2,2,\sim}^2].
$$

Collecting all the moment estimates for $\xi_y = 0$, we get

$$
\parallel P_J \hat{R}(0) \parallel_2^2 \le c\big[\eta \parallel P_J R \parallel_{2,2}^2, + \frac{1}{\eta\varepsilon^2} \parallel (I-P_J)R \parallel_{2,2}^2 + \parallel g \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel \Psi \parallel_{2,2,\sim}^2 \big]
$$

and this concludes the proof of Lemma 4.4.

□

Combining the two steps we can prove

Theorem 4.5. *Assume $\delta^2 = \varepsilon\gamma$. Then, for γ small enough, if $\int Mgdv = 0$, the solution of (4.33) satisfies*

$$\| P_J R \|_{2,2}^2 \leq c\big(\| (I-P_J)g \|_{2,2}^2 + \frac{1}{\varepsilon^2} \| P_J g \|_{2,2}^2 + \frac{1}{\varepsilon^4} \| \Psi \|_{2,2,\sim}^2 \big) \tag{4.49}$$

$$\| \nu^{\frac{1}{2}}(I-P_J)R \|_{2,2}^2 \leq c \| P_J g \|_{2,2}^2 + \frac{1}{\varepsilon^2\eta} \| \Psi \|_{2,2,\sim}^2 + \varepsilon^2 \| (I-P_J)g \|_{2,2}^2$$

Proof. Replace (4.22) in (4.38), and use γ and η small. □

The step from L_2 to L_∞ is done by studying the solution along the characteristics as [1]. The result is

$$\| \nu^{\frac{1}{2}} R \|_{\infty,2}^2 \leq c\Big(\frac{1}{\varepsilon^2} \| R \|_{2,2}^2 + \varepsilon^2 \| \nu^{-\frac{1}{2}} g \|_{\infty,2}^2 + \frac{1}{\varepsilon^2} \| \bar{\Psi} \|_{2,2,\sim}^2 \Big).$$

Then, by Theorem 4.5,

$$\| \nu^{\frac{1}{2}} R \|_{\infty,2}^2 \leq c\Big(\frac{1}{\varepsilon^2} \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2}^2 + \frac{1}{\varepsilon^4} \| P_J g \|_{2,2}^2 + \varepsilon^2 \| \nu^{-\frac{1}{2}} g \|_{\infty,2}^2 + \frac{1}{\varepsilon^6} \| \bar{\Psi} \|_{2,2,\sim}^2 \Big). \tag{4.50}$$

With the a priori estimates provided by Theorem 4.5, we are now in the position of proving the existence theorem for the remainder equation (4.16-4.18), by an iteration procedure.

Theorem 4.6. *There exists a solution in $L_2([-\pi,\pi]\times\mathbb{R}^3; Mdvdy)$ to the problem (4.10-4.13) and is unique.*

Proof. The remainder term R will be obtained as the limit of the approximating sequence R^n, where $R^0 = 0$ and

$$v_y \frac{\partial R^{n+1}}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2}\mu(y) v_x \big(v_x \frac{\partial R^{n+1}}{\partial v_y} - v_y \frac{\partial R^{n+1}}{\partial v_x}\big) = \frac{1}{\varepsilon}[L_J R^{n+1} + H_1 R^{n+1}] + J(R^n, R^n) + a, \tag{4.51}$$

$$\begin{aligned} R^{n+1}(-\pi, v) &= \frac{M_-}{M}\int_{-w_y>0} (R^{n+1}(-\pi,w) + \frac{1}{\varepsilon}\Psi(-\pi,w))|w_y| M dw, \\ &\quad -\frac{1}{\varepsilon}\Psi(-\pi,w), \quad v_y > 0, \\ R^{n+1}(\pi, v) &= \frac{M_+}{M}\int_{w_y>0} \frac{1}{\varepsilon}\Psi(\pi,w)|w_y| M dw - \frac{1}{\varepsilon}\Psi(\pi,v), \quad v_y < 0. \end{aligned} \tag{4.52}$$

Here a is of order ε^4 and it holds that R^{n+1} satisfies also the boundary conditions (4.18). The function R^1 is solution to (4.33-4.35) with $g = a$. Then, by using the a priori estimates of Theorem 4.5 and (4.50), together with the exponential decrease of Ψ, we obtain, for some constant c_1,

$$\| \nu^{\frac{1}{2}} R^1 \|_{\infty,2} \leq c_1\varepsilon^2, \quad \| \nu^{\frac{1}{2}} R^1 \|_{2,2} \leq c_1\varepsilon^3.$$

By induction for ε sufficiently small,

$$\begin{aligned}&\| \nu^{\frac{1}{2}} R^j \|_{\infty,2} \leq 2c_1\varepsilon^2, \quad j \leq n+1,\\ &\| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2} \leq c_2\varepsilon^{\frac{3}{2}} \| \nu^{\frac{1}{2}} (R^n - R^{n-1}) \|_{2,2}, \quad n \geq 1,\end{aligned}$$

for some constant c_2. Namely,

$$\begin{aligned}&v_y \frac{\partial R^{n+1}}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2}\mu(y)v_x(v_x \frac{\partial (R^{n+2}-R^{n+1})}{\partial v_y} - v_y \frac{\partial (R^{n+2}-R^{n+1})}{\partial v_x}))\\ &= \frac{1}{\varepsilon^2} L_J(R^{n+2}-R^{n+1}) + \frac{1}{\varepsilon} H_1(R^{n+2}-R^{n+1}) + \frac{1}{\varepsilon} G^{n+1},\\ &(R^{n+2}-R^{n+1})(-\pi,v) = \frac{M_{\mp}}{M} \int_{w_z \lesseqgtr 0} (R^{n+2}-R^{n+1})(-\pi,w)|w_y|M_- dw,\ v_y > 0,\\ &(R^{n+2}-R^{n+1})(\pi,v) = 0,\ v_y < 0\end{aligned}$$

Here, $G^{n+1} = (I-P)G^{n+1} = J(R^{n+1}+R^n, R^{n+1}-R^n)$. It follows that

$$\begin{aligned}\| \nu^{\frac{1}{2}} (R^{n+2}-R^{n+1}) \|_{2,2} &\leq c\varepsilon^{-\frac{1}{2}} \| \nu^{-\frac{1}{2}} G^{n+1} \|_{2,2}\\ &\leq c\varepsilon^{-\frac{1}{2}} \Big(\| \nu^{\frac{1}{2}} R^{n+1} \|_{\infty,2} + \| \nu^{\frac{1}{2}} R^n \|_{\infty,2} \Big) \| \nu^{\frac{1}{2}} (R^{n+1}-R^n) \|_{2,2}\\ &\leq c_2\varepsilon^{\frac{3}{2}} \| \nu^{\frac{1}{2}} (R^{n+1}-R^n) \|_{2,2} .\end{aligned}$$

Consequently,

$$\begin{aligned}\| \nu^{\frac{1}{2}} R^{n+2} \|_{2,2} \quad &\leq \| \nu^{\frac{1}{2}} (R^{n+2}-R^{n+1}) \|_{2,2} + ... + \| \nu^{\frac{1}{2}} (R^2-R^1) \|_{2,2}\\ &+ \| \nu^{\frac{1}{2}} R^1 \|_{2,2} \leq 2c_1\varepsilon^3,\end{aligned}$$

for ε small enough. Similarly, $\| R^{n+2} \|_{\infty,2} \leq 2c_1\varepsilon^2$. In particular $\{R^n\}$ is a Cauchy sequence in $L^2_M([-\pi,\pi]^2 \times \mathbb{R}^3)$. The existence of a solution R to (4.10) follows. Uniqueness follows along the same path. □

Corollary 4.7. *There exists an isolated L_2-solution F to (1.23) such that*

$$\| M^{-1}[F - M_\delta] \|_{2,2} \leq c\varepsilon.$$

Proof of Theorem 1.1. The positivity can be proved as in [3]–[6]. Then, by Corollary 2.2 and 4.7 we have, for $q = 2, \infty$

$$\begin{aligned}\| M^{-1}[F - M(1,1,(\delta U,0,0))] \|_{q,2} &\leq \| M^{-1}[F - M_\delta] \|_{q,2}\\ &+ \| M^{-1}[M_\delta - M(1,1,(\delta U,0,0))] \|_{q,2} \leq c(\varepsilon + \delta^2)\end{aligned}$$

which implies (1.25) by taking into account the relation $\delta^2 = \gamma\varepsilon$. □

Acknowledgments. R. E. and R. M. are partially supported by Murst-Prin. The work of A. N. has been supported by the ANR EGYPT. L. A. and A. N. thank the University of L'Aquila and R. E. and R. M. thank the Chalmers University and the IHES for the kind hospitality.

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