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**L. Arkeryd · R. Esposito · R. Marra ·
A. Nouri**

Stability for Rayleigh-Benard convective solutions of the Boltzmann equation.

Abstract We consider the Boltzmann equation for a gas in a horizontal slab, subject to a gravitational force. The boundary conditions are of diffusive type, specifying the wall temperatures, so that the top temperature is lower than the bottom one (Benard setup). We consider a 2-dimensional convective stationary solution, which is close for small Knudsen number to the convective stationary solution of the Oberbeck-Boussinesq equations, near above the bifurcation point, and prove its stability under 2-d small perturbations, for Rayleigh number above and close to the bifurcation point and for small Knudsen number.

Keywords: Boltzmann equation, Benard problem, stability

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L. Arkeryd
Chalmers Institute of Technology, Gothenburg, Sweden. E-mail: arkeryd@math.chalmers.se

R. Esposito
Dipartimento di Matematica pura ed applicata - Università di L'Aquila, Coppito - 67100 L'Aquila - Italy
E-mail: esposito@axp.mat.uniroma2.it

R. Marra
Dipartimento di Fisica and Unità INFN, Università di Roma Tor Vergata, 00133 Roma, Italy
E-mail: marra@roma2.infn.it

A. Nouri
LATP, Université d'Aix-Marseille I, Marseille, France.E-mail: nouri@cmi.univ-mrs.fr

1 Introduction

We study the small Knudsen number solutions to the Boltzmann equation in a slab with diffusive boundary conditions in the presence of a gravitational field.

In a previous paper [1] we have proved the existence and stability globally in time for small Knudsen number of a positive one-dimensional stationary solution to the Boltzmann equation, which is close to the hydrodynamic laminar solution of the Oberbeck-Boussinesq (O-B) equations. At the hydrodynamical level there is a bifurcation phenomenon: when the vertical temperature gradient exceeds a certain critical value, the laminar one-dimensional solution loses stability and various two- or three-dimensional pattern flows appear. In particular, it has been proved the existence of a two-dimensional roll solution of the O-B equations close to the bifurcation point. Its stability under suitable perturbations has also been proved.

In this paper we construct, by means of perturbative arguments (expansion method), for small Knudsen number, a positive two dimensional solution to the stationary Boltzmann equation, which is close to this roll solution. Moreover, we prove its stability for long times under a suitable class of two dimensional initial perturbations. These results are true for values of the Rayleigh number above and close to the bifurcation value, provided that the force is small enough. To state our result, we need to introduce some notation.

Consider a gas in a 2-dimensional box of height $2\pi d$ and length $2\pi h$, under the action of a gravitational force g in the direction z. The upper and lower walls are kept at temperature T_+ and T_- respectively, with $T_+ < T_-$, with no-slip conditions, while periodicity is assumed in the horizontal direction. At the kinetic level, the behavior of the gas is given by the following Boltzmann equation with boundary conditions diffusive in the z direction and periodic in the x direction, written in dimensionless form,

$$\begin{aligned}
&\frac{\partial F}{\partial t}+\frac{1}{\varepsilon}\,v_x\frac{\partial F}{\partial x}+\frac{1}{\varepsilon}v_z\frac{\partial F}{\partial z}-G\frac{\partial F}{\partial v_z}=\frac{1}{\varepsilon^2}Q(F,F),\\
&F(0,x,z,v)=F_0(x,z,v),\quad (x,z)\in(-\mu\pi,\mu\pi)\times(-\pi,\pi),\ v\in\mathbb{R}^3,\\
&F(t,x,\mp\pi,v)\ =M_\mp(v)\int_{w_z\lesseqgtr 0}|w_z|F(t,x,\mp\pi,w)dw, t>0,\quad v_z\gtreqless 0,
\end{aligned}\tag{1.1}$$

for $x\in[-\mu\pi,\mu\pi]$, where

$$F_0\ge 0,\quad M_-=\frac{1}{2\pi}e^{-\frac{v^2}{2}},\quad M_+(v)=\frac{1}{2\pi(1-2\pi\varepsilon\lambda)^2}e^{-\frac{v^2}{2(1-2\pi\varepsilon\lambda)}},$$

$$\varepsilon=\frac{\ell_0}{d},\quad G=\frac{1}{\varepsilon}\frac{dg}{2T_-},\quad \lambda=\frac{1}{\varepsilon}\frac{T_--T_+}{2\pi T_-},\quad \mu=\frac{h}{d},$$

$$Q(f,g)(z,v,t)=\frac{1}{2}\int_{\mathbb{R}^3}dv_*\int_{S_2}d\omega B(\omega,|v-v_*|)\big\{f'_*g'+f'g'_*-f_*g-g_*f\big\}.$$

Here h',h'_*,h,h_* stand for $h(x,z,v',t),h(x,z,v'_*,t),h(x,z,v,t),h(x,z,v_*,t)$ respectively, $S_2=\{\omega\in\mathbb{R}^3\,|\omega^2=1\}$, B is the differential cross section

$2B(\omega, V) = |V \cdot \omega|$ corresponding to hard spheres, and v, v_* and v',v'_* are precollisional and postcollisional velocities or conversely. The boundary conditions are such that the condition of impermeability of the walls,

$$\int_{\mathbb{R}^3} dv F v_z dv = 0, \tag{1.2}$$

is satisfied. The solution depends on the parameter $\varepsilon = 2\sqrt{6}Kn/\sqrt{5\pi}$, where Kn is the Knudsen number given in terms of ℓ_0, the mean free path of the gas in equilibrium at temperature T_- and density $\bar{\rho}$. We have also put $Ma = \varepsilon\sqrt{6}/\sqrt{5}$, where Ma is the Mach number. With this choice, the Rayleigh number ([19]) is $Ra = \dfrac{16G(2\pi\lambda)}{\pi}$, independent of ε. We will fix through the paper the parameter G such that $G \le G_0$ with G_0 suitably small. Fix $h = \dfrac{2\pi d}{\alpha_c}$ where α_c is the critical wave number for the first bifurcation. The linear analysis of the O-B equations with rigid-rigid boundary conditions, (3.1) below, describing the behavior of the fluid at hydrodynamic level, gives a critical value α_c and a corresponding critical Rayleigh number Ra_c [7]. The parameter λ will be chosen in an interval $[\lambda_c, (1+\delta)\lambda_c]$, for δ small and with λ_c determined by the condition that $Ra_c = \dfrac{16G(2\pi\lambda_c)}{\pi}$ is the critical value. With these choices, at the hydrodynamical level, two roll solutions will appear at the bifurcation point, consisting of one roll, rotating clockwise and anticlockwise respectively. These solutions are constructed perturbatively for δ small in a rigorous way [12] and their local non-linear stability has been proved for small initial perturbations with the same period of the roll solution [12].

The clockwise solution h_s is then of the form

$$h_s = h_\ell + \delta\ h_{con} + O(\delta^2), \tag{1.3}$$

where h_ℓ is the laminar solution and h_{con} is the eigenfunction corresponding to the least eigenvalue of the linearized Boussinesq problem around the laminar solution (see Section 4 for the precise definition).

In this paper, we construct a stationary solution F_s of the Boltzmann equation, which is close for ε small to the hydrodynamical solution (say, the clockwise one) in the sense that it can be written as a truncated expansion in ε, $F_s = M + \varepsilon f_s + O(\varepsilon^2)$ with $M = \dfrac{1}{(2\pi)^{3/2}} e^{-\frac{v^2}{2}}$ and

$$f_s = M\Big(\rho_s + u_s \cdot v + T_s \frac{|v|^2 - 3}{2}\Big)\ ,$$

where ρ_s, u_s, T_s are expressed in terms of h_s. Moreover, we prove the kinetic non linear stability of F_s under suitable initial perturbations.

We study the Boltzmann equation for the perturbation $\Phi = M^{-1}(F - F_s)$ with the initial datum

$$\Phi_0(x, z, v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(0, x, z, v) + \varepsilon^5 p_5 \tag{1.4}$$

where $\Phi^{(n)}(0,x,z,v)$ is the n-th term of the expansion introduced in the next paragraph, computed at time $t=0$, and the ϵ-dependent p_5 is arbitrary but for having total mass $\int dvdxdzMp_5 = 0$ and satisfying (3.8).

We write also the time dependent solution in terms of a truncated expansion in ε

$$\Phi(t,x,z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(t,x,z,v) + \varepsilon R(t,x,z,v), \quad (x,z) \in \Omega_\mu, \qquad (1.5)$$

where $\Omega_\mu = [-\mu\pi, \mu\pi] \times [-\pi, \pi]$. The first term of the expansion in ε is

$$\Phi^{(1)} = \rho^1 + u^1 \cdot v + \theta^1 \frac{|v|^2 - 3}{2} \ ,$$

where the fields $\rho^1(t,x,z), u^1(t,x,z), \theta^1(t,x,z)$ are solutions of the hydrodynamic equations for the perturbation, with initial datum (u^1_0, θ^1_0). The initial data are chosen as follows: let (u^1_0, θ^1_0) be an initial perturbations of the convective solution (u_s, θ_s) sufficiently small to ensure that the solution $(u(t,x,z), \theta(t,x,z)) = (u_s(x,z) + u^1(t,x,z), \theta_s(x,z) + \theta^1(t,x,z))$ of the initial boundary value problem for the O-B (3.1) equations exists globally in time and converges to (u_s, θ_s) as $t \to +\infty$.

The next orders involve kinetic corrections in the bulk as well as boundary layer corrections. In fact, at next orders, the standard Hilbert expansion bulk terms do not satisfy the diffusive boundary conditions and boundary layer correction terms are to be included to restore the boundary conditions. They are computed solving suitable Milne problems in the presence of a force $\overrightarrow{F}$ and a source term S. Under suitable assumptions this problem has z-smooth solutions when $\overrightarrow{F}$ is a potential force decaying fast enough at infinity [6]. We give in Section 3 a procedure to compute the Φ_n's in the time dependent case and show that they have good enough properties as consequence of the smoothness of the hydrodynamic solution. In particular, they inherit the smallness and decay properties of the hydrodynamic solution, such as the exponential decay in time. The main difficulty is then the control of the remainder R asymptotically in time. The equation for the remainder R is a weakly non-linear Boltzmann equation with a source B, generated by the terms of the expansion. Since it is weakly non-linear, it is enough to get good estimates for the associated linear problem, which is of the form

$$\frac{\partial}{\partial t}R + \frac{1}{\varepsilon}(v_x \frac{\partial}{\partial x}R + v_z \frac{\partial}{\partial z}R) - M^{-1}G\frac{\partial}{\partial v_z}(MR) = \frac{1}{\varepsilon^2}LR + \frac{1}{\varepsilon}J(\Phi_H, R) + B,$$

where L is the linearized Boltzmann operator, defined in Section 2, and $J(\Phi_H, R)$ is a linear operator depending on the perturbed stationary solution and on the first terms of the expansion. The operator L is non positive on $L_2(\mathbb{R}^3, Mdv)$, but has a non trivial null space $\texttt{Kern}(L)$. The linear operator $J(\Phi_H, R)$ is at least of order ε but it is the main contribution for $R \in \texttt{Kern}(L)$. The control of the component of R in the orthogonal to $\texttt{Kern}(L)$ is given by the well known spectral inequality for L (see e.g. [17]),

$$-(f, Lf) \geq C((1-P)f, \nu(1-P)f), \qquad (1.6)$$

where (f, Lf) is the scalar product in $L_2(\mathbb{R}^3, Mdv)$, P is the projector on $\mathtt{Kern}(L)$, $\nu(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}_2} d\omega \frac{1}{2}|(v - v_*) \cdot \omega| M(v_*)$. On the other hand, it is easy to check that

$$(f, J(\Phi_H, Pf)) \leq C\|\nu^{1/2} Pf\| \, \|\nu^{1/2}(1 - P)f\|.$$

Using this (and ignoring the boundary contributions) one gets the differential inequality

$$\frac{1}{2}\frac{d}{dt} \parallel f \parallel_{2,2}^2 \leq C \parallel f \parallel_{2,2}^2 + \int_{\Omega_\mu} |(B, f)|,$$

with $\| \cdot \|_{2,2}$ the norm in $L_2(\Omega_\mu \times \mathbb{R}^3, Mdxdzdv)$. This produces bounds growing exponentially in time. To avoid this we use a spectral inequality for the operator
$L_J(f) = Lf + \varepsilon J(\Phi_H, Pf)$. The inclusion of the second term in the new operator simplifies all the arguments in the control of the remainder. The price to be paid is that L_J is not self-adjoint and its null space is more complicated. This inequality is an important new ingredient for proving stability results in the Boltzmann equation framework. In [2] there was no need to use this inequality to get the global stability result because in that paper it was possible to take advantage from the fact that the constant C in the previous inequality can be taken small, while the parameter controlling the bifurcation can grow beyond the bifurcation value. In the Rayleigh-Benard setting, there is not such freedom.

To study the hydrodynamic part of R, PR, we use a duality argument which was introduced by N. Maslova [17]. To illustrate how it works, let us consider the typical equation one has to study in the stationary case

$$v \cdot \nabla_x R - \varepsilon G \frac{1}{M} \frac{\partial}{\partial v_z}(MR) = \frac{1}{\varepsilon} L_J R + q$$

with prescribed incoming data at $z = \pm\pi$, periodic in x. One also considers the dual equation

$$v \cdot \nabla_x \phi - \varepsilon G \frac{1}{M} \frac{\partial}{\partial v_z}(M\phi) = \frac{1}{\varepsilon} L_J^* \phi + h,$$

with vanishing incoming data at $z = \pm\pi$, periodic in x. By taking the inner product of the first by ϕ and the second by R and summing, one gets an estimate of $\|h\|$ in terms of q and ϕ with suitably small coefficients. For that, we need G small, but we can anyway have large Rayleigh number by increasing λ. Then one chooses $h = PR$ and thus shows that $\|PR\|$ can be bounded in terms of $\|\phi\|$ and $\|q\|$. The equation for ϕ is studied by means of Fourier analysis, which provides a good control on ϕ but for the 0-moment. The estimate of the 0-moment is based on a direct approach, by means of rather explicit calculations of the first few moments of the solution using o.d.e. analysis, for the one-dimensional case. This allows a reasonably simple analysis when the incoming data are prescribed. Dealing with diffuse reflection requires more technical steps. This method has been exploited in [2] and has been extended in [1] to the one-dimensional Benard problem, which is more difficult to deal with because of the presence of the force and the diffusive boundary conditions. In this paper, we extend the one-dimensional analysis to the two-dimensional case by using that, both in the conductive and convective case, for $\mathcal{R} < \mathcal{R}_c(1+\delta)$ and δ small, the contributions due to the inhomogeneity in the x variable are small and can be included perturbatively. A by-product of this analysis is the extension of the result in [1] to two-dimensional initial perturbations.

Finally, the control of the nonlinearity requires L^∞-estimates (in space), which are more intricate by the presence of the force. We use techniques based on the study of the characteristics, which are no more straight lines because of the presence of the force. This is controlled by looking at the characteristics in the phase space like in [9] and [1].

The main result of this paper is summarized in the following theorem.

Theorem 1.1 *Assume that the gravitational force is such that $G \le G_0$, with G_0 small enough, and $\lambda \in [\lambda_c, (1+\delta)\lambda_c]$. Then, there are δ_0 and ε_0 small enough such that for $\delta \le \delta_0$, there exists a positive steady solution F_s to the Boltzmann equation, such that for $\varepsilon \le \varepsilon_0$,*

$$\| M^{-1}[F_s - (M + \varepsilon f_s)] \|_{2,2} \le c\varepsilon^2$$

Assume also that the initial perturbation matches the expansion up to order ε^4 as detailed in Section 3 below, and is small as detailed in Section 4. For such perturbations the stationary solution is stable uniformly in ε for $\varepsilon \le \varepsilon_0$.

Here, stable means that the perturbation vanishes asymptotically in time. This is a consequence of the inequality

$$\int_0^{+\infty} dt \int_{\Omega_\mu} dxdz \int_{\mathbb{R}^3} dv |\Phi(t,x,,z,v)|^2 M(v) < \infty \ , \tag{1.7}$$

which is proved in Section 4, and of the regularity of the solution which follows from our construction.

The positivity of the stationary solution is obtained by using the methods in [5]. We remark that the method presented here for proving stability strongly relies on the fact that the problem we are dealing with has suitable stability properties at the fluid dynamic level, which we show to be preserved in the kinetic setup by means of a perturbative analysis starting from an Hilbert-type asymptotic expansion plus boundary layer corrections. The preservation of the fluid dynamic stability at kinetic level also occurs in the Taylor-Couette case discussed in [2], where the bifurcation phenomenon also arises.

The paper is organized as follows: in Section 2 we construct the stationary solution as a Hilbert asymptotic series in ε. For sake of shortness, we choose not to give here the construction of the terms of the expansion, and refer for that to Section 3, where we show explicitly the analogous construction in the time-dependent case. We do complete the proof of the existence of the solution in the stationary case in Section 2, by proving the main theorem on the remainder. In Section 4 we deal with the remainder in the time-dependent case and prove the stability result.

2 Stationary case

In this section the stationary case is treated. Here, (and also in Section 4), for sake of simplicity, we consider a square box $[-\pi,\pi]^2$ instead of a rectangular box $[-\mu\pi,\mu\pi]\times[-\pi,\pi]$ and hence the x-derivative in the equation will have a factor μ in front.

We write $F_s = M(1+\Phi_s^\varepsilon)$ in terms of a truncated expansion in the Knudsen number ε plus a rest term:

$$\Phi_s^\varepsilon(x,z,v) = \sum_1^5 \varepsilon^j \Phi_s^{(j)}(x,z,v) + \varepsilon R_{s,\varepsilon}(x,z,v).$$

The expansion will not be given explicitly here since all the ideas and details are in the previous paper [1]. We only remark that the construction of the $\Phi_s^{(j)}$'s relies on the solution to a Milne problem with external force given in [6]. Moreover, we will give in the next sections more details on the analogous expansion in the time dependent case. In this section we study the equation for the remainder, beginning with some results on linear existence together with corresponding a priori estimates. The section ends with an existence proof for the nonlinear stationary rest term.

We denote by $H = L^2_M(\mathbb{R}^3)$ the Hilbert space of the measurable functions on $\mathbb{R}^3$ with inner product $(\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\, M\cdot)_2$, where $(\,\cdot\,,\,\cdot\,)_2$ is the standard L^2 inner product and $\|\cdot\|_2$ the standard L^2-norm, while $\|\cdot\|$ is the norm corresponding to $(\,\cdot\,,\,\cdot\,)$.

The linearized Boltzmann operator is defined, for any f in a dense subset of H as:

$$Lf = 2M^{-1}Q(M,Mf). \tag{2.1}$$

It is well know that it can be decomposed as $L = -\nu I + K$, where I is the identity, K a compact operator and $\nu(v)$, defined in the Introduction, is a smooth function satisfying the estimates $\nu_0(1+|v|) \le \nu(v) \le \nu_1(1+|v|)$ for some positive ν_0 and ν_1. The operator L is a non positive self-adjoint operator with domain $\mathcal{D}_L = \{f \in H \mid \|\nu^{\frac{1}{2}} f\| < +\infty\}$.

The functions $\psi_0 = 1, \psi_1 = \psi_x = v_x, \psi_2 = \psi_y = v_y, \psi_3 = \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2-3)$ form an orthonormal basis for the kernel of L in H, $\texttt{Kern}(L)$. For any function in H introduce the orthogonal splitting $f = f_\| + f_\perp := Pf + (I-P)f$, where $f_\|$ is called fluid dynamic part and is given by

$$f_\|(x,z,v) = \sum_{j=0}^4 f_j\psi_j, \quad f_j := (f,\psi_j), \quad j=,0,\dots,4,$$

while the non hydrodynamic part $f_\perp$ satisfies $(f_\perp,\psi_j) = 0, \quad j=,0,\dots,4$. P denotes the projector from H on $\texttt{Kern}(L)$. Note that the range of L is $(I-P)H$. We will use the same symbol for the projections also when dealing with functions depending on x,z,t. We remind that the operator L satisfies the spectral inequality (1.6).

For $1 \le q \le +\infty$, let $\tilde L^q$ be the Banach space of the measurable functions from $[-\pi,\pi]^2$ in H, identified with the space of measurable functions from $[-\pi,\pi]^2 \times \mathbb{R}^3$ in $\mathbb{R}$ with norm

$$\| f\|_{q,2} = \left(\int_{\mathbb{R}^3} dv M(v) \left(\int_{[-\pi,\pi]^2} dxdz |f(x,z,v)|^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}},$$

that is

$$\tilde L^q =:= \{ f : [-\pi,\pi]^2 \times \mathbb{R}^3 \to \mathbb{R} \,|\, \| f \|_{q,2} < +\infty \}.$$

Moreover $(\,\cdot\,,\,\cdot\,)_{2,2}$ is the corresponding inner product for $q=2$.

We also need a function space for the boundary functions. We denote by $\mathbb{R}^3_\pm$ the sets $v=(v_x,v_y,v_z)$ such that $v_z \gtrless 0$. We consider the functions on $[-\pi,\pi]\times\{-\pi\}\times\mathbb{R}^3_+ \cup [-\pi,\pi]\times\{\pi\}\times\mathbb{R}^3_-$ and define the norm

$$\| f \|_{q,2,\sim} = \sup_{\pm} \left(\int_{\mathbb{R}^3_\pm} dv |v_z| M(v) \left(\int_{[-\pi,\pi]} dx |f(x,\mp\pi,v)|^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}.$$

The Banach space L^+ is the set of such functions with finite $\|\cdot\|_{2,2,\sim}$ norm. The ingoing and outgoing trace operators $\gamma^\pm$ are defined by

$$\gamma^\pm f = \begin{cases} f|_{z=-\pi}, & \text{if } v \in \mathbb{R}^3_\pm, \\ f|_{z=\pi}, & \text{if } v \in \mathbb{R}^3_\mp. \end{cases}$$

The function space where the stationary solution will be constructed is the space

$$\mathcal{W}^{q,-} := \{ f : [-\pi,\pi]^2 \times \mathbb{R}^3 \to \mathbb{R} \,|\, \nu^{\frac{1}{2}} f \in \tilde L^q,\ \nu^{-\frac{1}{2}} Df \in \tilde L^q,\ \gamma^+ f \in L^+ \}.$$

Note that the norm $\|\cdot\|_{2,2,\sim}$ is defined only for incoming velocities. In the sequel, with an abuse of notation we will denote by $\| \gamma^- f \|_{2,2,\sim}$ the $\|\cdot\|_{2,2,\sim}$-norm of $S\gamma^- f$, where S is the reflection of the z component of the velocity.

We do not explain here how to construct the terms of the expansion $\Phi_z^{(j)}$. We simply state a theorem about their properties. We assume that the Rayleigh number Ra is in $(Ra_c, Ra_c+\delta)$ with $\delta>0$ and sufficiently small and will consider, for sake of definiteness, the clockwise convective solution corresponding to it.

Theorem 2.1 *The functions $\Phi_s^{(n)}$, $n=1,\dots,5$ and $\psi_{n,\varepsilon}$ can be determined so as to satisfy the boundary conditions*

$$\begin{aligned} \Phi^{(n)}(x,\mp\pi,v) &= \frac{M_\mp(v)}{M(v)} \int_{w_z \lessgtr 0} |w_z| M[\Phi^{(n)}(x,\mp\pi,w) - \psi_{n,\varepsilon}(x,\mp\pi,w)]dw \\ &\quad + \psi_{n,\varepsilon}(x,\mp\pi,w),\ t>0,\ v_z \gtrless 0, \end{aligned}$$

and the normalization condition $\int_{\mathbb{R}^3\times[-\pi,\pi]^2} dvdxdz\Phi^{(n)} = 0$, *so that the asymptotic expansion in* ε *for the stationary problem (1.1), truncated to the order* 5 *is given by*

$$F_s^{(exp)}(x,z,v) = M(v)\left(1+\sum_{n=1}^{5}\varepsilon^5\Phi^{(n)}(x,z,v)\right).$$

The functions $\Phi^{(n)}$*'s satisfy the conditions*

$$\parallel \Phi^{(n)} \parallel_{2,2}< \infty, \quad \parallel \Phi^{(n)} \parallel_{\infty,2}< \infty \ , \quad n=1,\dots,5.$$

Moreover the $\Phi^{(n)}$*'s differ from those of the laminar solution by* $O(\delta)$.
The functions $\psi_{n,\varepsilon}$ *are such that* $\|\psi_{n,\varepsilon}\|_{q,2,\sim}$, $q=2,\infty$ *are exponentially small as* $\varepsilon\to 0$ *and* $\int_{\mathbb{R}^3} dvv_zM(v)\psi_{n,\varepsilon}=0$. *Finally, there exists a stationary solution to (1.1) in the form*

$$F_s = F_s^{(exp)} + \varepsilon R_{s,\varepsilon}.$$

The remainder $R_{s,\varepsilon}$, *simply denoted by* R *solves the boundary value problem*

$$\begin{aligned}&\mu v_x\frac{\partial}{\partial x}R + v_z\frac{\partial}{\partial z}R - \varepsilon G M^{-1}\frac{\partial (MR)}{\partial v_z}\\ &= \frac{1}{\varepsilon}LR + \sum_{i=1}^{5}\varepsilon^{j-1}J(\Phi^{(j)},R) + J(R,R) + A \end{aligned}\tag{2.2}$$

$$\begin{aligned}R(x,\mp\pi,v) = &\frac{M_\mp(v)}{M(v)}\int_{w_z\lesseqgtr 0}|w_z|M(w)\Big(R(x,\mp\pi,w)+\frac{1}{\varepsilon}\bar\psi_\varepsilon(x,\mp\pi,w)\Big)dw\\ - &\frac{1}{\varepsilon}\bar\psi_\varepsilon(x,\mp\pi,v), \qquad \textit{for } v_z\gtrless 0 \quad \textit{and } x\in[-\pi,\pi]\end{aligned}\tag{2.3}$$

where $\frac{1}{\varepsilon}\bar\psi_\varepsilon = -\sum_{n=1}^{5}\varepsilon^n\psi_{n,\varepsilon}$, $J(h,g)=\dfrac{2}{M}Q(Mh,Mg)$ *and* A *is a smooth function bounded in* $\|\cdot\|_{q,2}$, $q=2,\infty$, *such that* $\displaystyle\int_{\mathbb{R}^3} dvM(v)A=0$.

We do not give here the explicit expression of A for which we refer to [1]. It has to be considered as a known term in the rest of this section as well as the $\Phi^{(j)}$'s. The terms of the expansions contribute to A together with their space and velocity derivatives. It suffices to know that it has finite $\|\cdot\|_{q,2}$-norm. Note that M and M_- differ just by the normalization and that the $\psi_{n,\varepsilon}$'s are known functions exponentially small in ε due to the boundary layer corrections in the expansion. With these boundary conditions the remainder satisfies the impermeability condition

$$\int_{\mathbb{R}^3} dvv_zMR = 0 \tag{2.4}$$

at the walls.

Now we start the analysis of the equation for the remainder R. The first important result is a spectral property for the operator L_J below: fix x, z and define, for each $f \in H$

$$L_J f = Lf + \varepsilon NPf \ , \tag{2.5}$$

with

$$N = J(q, \cdot), \quad q = \sum_{n=1}^{5} \varepsilon^{n-1} \Phi_s^{(n)}.$$

Note that in the time-dependent case the form of q will be slightly different but without affecting the argument below.

The operator L_J is not symmetric in H. We denote by L_J^* its adjoint. To characterize the kernel of L_J, $\mathtt{Kern}(L_J)$, note that the functions

$$\bar{\psi}_j = \psi_j - \varepsilon L^{-1} N \psi_j, \tag{2.6}$$

are in $\mathtt{Kern}(L_J)$, where, for $\in (I-P)H$, $L^{-1}f$ denotes the unique solution of $Lg = f$ orthogonal to $\mathtt{Kern}(L)$. In fact, $L_J \bar{\psi}_j = L\psi_j - \varepsilon NP\psi_j + \varepsilon NP\psi_j - \varepsilon^2 NP[L^{-1}NP\psi_j] = 0$, because ψ_j are in $\mathtt{Kern}(L)$ and the last term is zero since P kills the terms in the range of L^{-1}.

It is easy to check, by using (1.6), that, at least for ε sufficiently small, they actually span $\mathtt{Kern}(L_J)$. Indeed, suppose that there is $g \in \mathtt{Kern}(L_J)$ with $\|g_\perp\| = 1$ such that $(g, \bar{\psi}_j) = 0$. This implies $g_\parallel = \varepsilon \sum_{j=0}^{4} (g_\perp, L^{-1}N\psi_j)\psi_j$. Moreover, $0 = (g, L_J g) = (g_\perp, Lg_\perp) + \varepsilon(g_\perp, Ng_\parallel)$. Therefore by (1.6), $C \le -(g_\perp, Lg_\perp) = \varepsilon^2 \sum_{j=0}^{4} (g_\perp, N\psi_j)(g_\perp, L^{-1}N\psi_j) \le \alpha\varepsilon^2$, for some positive α independent of ε. This is in contradiction with $C > 0$ for ε sufficiently small.

Let P_J denote the orthogonal projection onto $\mathtt{Kern}(L_J)$. Since the range of L_J is $(I-P)H$, it follows that $\mathtt{Kern}(L_J^*) = PH = \mathtt{Kern}(L)$.

Consider the space $\mathcal{N}$ generated by $\{\psi_0, \dots, \psi_4, L^{-1}N\psi_0, \dots, L^{-1}N\psi_4\}$. We decompose this space into $\mathtt{Kern}(L_J)$ and its orthogonal complement in $\mathcal{N}$, L_1.

Moreover, $(I-P)H$ can be decomposed in the span of $\{N\psi_1, \dots, N\psi_4\}$ and its orthogonal complement, $\bar{L}_o$. It is easy to check that

$$H = \mathtt{Kern}(L_J) \oplus L_1 \oplus \bar{L}_o.$$

Indeed it is enough to show that $N\psi_\ell, \ell = 0, \dots, 4$ is a combination of $L^{-1}N\psi_j$'s and $u \in \bar{L}_o$:

$$N\psi_\ell = \sum_{j=0}^{4} \alpha_{\ell,j} L^{-1} N\psi_j + u, \quad (N\psi_k, u) = 0, \quad \ell, k = 0, \dots, 4.$$

We take the inner product with $N\psi_k$

$$(N\psi_k, N\psi_\ell) = \sum_{j=0}^{4} \alpha_{\ell,j} (N\psi_j, L^{-1}N\psi_k).$$

The matrix with elements $(N\psi_j, L^{-1}N\psi_k)$ is non singular, hence the $\alpha_{\ell,j}$ are uniquely determined, and u as a consequence.

Proposition 2.1 (Spectral gap property of L_J) *There is $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there is c independent of ε, (x, z) and δ, for which the following inequalities hold:*

$$-(L_J\varphi, \varphi) \geq c(\nu(I - P_J)\varphi, (I - P_J)\varphi), \tag{2.7}$$

$$-(L_J^*\varphi, \varphi) \geq c(\nu(I - P)\varphi, (I - P)\varphi) \tag{2.8}$$

Proof. First take (x, z) and the Rayleigh number fixed. It is then enough to consider the set $\mathcal{H}$ of all functions $\varphi = (I - P_J)\varphi = a\varphi_o + b\varphi_1$, where $\varphi_1 \in L_1$ and $\varphi_o = (I - P)\varphi_o \in \bar{L}_o$, and with $\nu^{\frac{1}{2}}\varphi_o$ and φ_1 of norm one and $a^2 + b^2 = 1$. First notice that $(\nu\varphi, \varphi)$ is uniformly bounded in $\mathcal{H}$. It remains to show that the left-hand side has a positive bound from below. But φ_1 can be decomposed as a sum of an element φ_{11} in $\mathtt{Kern}(L)$ with norm of order ε, and an element $L^{-1}N\varphi_{12}$ in the span of $\{L^{-1}N\psi_0, ..., L^{-1}N\psi_4\}$. Then

$$\begin{aligned}
&-(L_J\varphi, \varphi) = -\Big(aL\varphi_o + bN\varphi_{12} + b\varepsilon N\varphi_{11}, a\varphi_o + b\varphi_{11} + bL^{-1}N\varphi_{12}\Big)\\
&= -a^2(L\varphi_o, \varphi_o) - b^2(N\varphi_{12}, L^{-1}N\varphi_{12}) - \varepsilon b^2(N\varphi_{11}, L^{-1}N\varphi_{12})\\
&\geq C(a^2(\nu\varphi_o, \varphi_o) + b^2(L^{-1}N\varphi_{12}, L^{-1}N\varphi_{12})) + \varepsilon b^2(N\varphi_{11}, L^{-1}N\varphi_{12}).
\end{aligned}$$

The first equality follows from the fact that φ_{11} is in $\mathtt{Kern}(L)$ and is orthogonal to the range of L; the second equality is due to the fact that $(\varphi_0, N\varphi_{12}) = 0 = (\varphi_0, N\varphi_{11})$ by the definition of $\bar{L}_o$; the bound (1.6) has been used to obtain the inequality. Since the last term is of order ε^2, it follows that for $\varepsilon > 0$ and small enough

$$-(L_J\varphi, \varphi) \geq \frac{C}{2}(a^2(\nu\varphi_\perp, \varphi_\perp) + b^2(L^{-1}N\varphi_{12}, L^{-1}N\varphi_{12})) \geq c$$

for some $c > 0$. The first inequality in the proposition then follows, since the constant depends continuously on (x, z) and the Rayleigh number is in a compact set. The second inequality is obtained by similar arguments. □

We only consider Rayleigh numbers in a suitably small neighbourhood of the first bifurcation point, and take G sufficiently small as specified in the sequel. Constants which, independently of the parameter ε, can be made sufficiently small for the purposes of the proofs, will generically be denoted η. We will first give estimates in the linear case. The argument is inspired by the approach in [17] and heavily relies on the study of the space Fourier transform of R. We use the following definition of Fourier transform: for any $\xi \in \mathbb{Z}$

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{[-\pi,\pi]} dx \mathrm{e}^{-i\xi x} f(x).$$

When we want to specify that the Fourier transform is taken with respect to the variable x we write $\mathcal{F}_x$. In the sequel, if f is a function of $(x, z) \in [-\pi, \pi]^2$, $\hat{f}(\xi_x, \xi_z) = (\mathcal{F}_x\mathcal{F}_z f)(\xi_x, \xi_z)$. Finally, if f is a function of x, z and v, we define $< f >$ as the zero order Fourier coefficient of f, that is

$$< f > := \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} f(x, z, v)dxdz, \quad a.a.\ v \in \mathbb{R}^3 \tag{2.9}$$

and $\tilde f := f - < f >$.

An important tool in the analysis is the Green inequality, which we will use extensively in the rest of the paper in various different situations.

Consider the linear boundary value problem

$$\mu v_x \frac{\partial f}{\partial x} + v_z \frac{\partial f}{\partial z} - \varepsilon G M^{-1} \frac{\partial (Mf)}{\partial v_z} = \frac{1}{\varepsilon} L_J f + g, \tag{2.10}$$

with $\int M g dv = 0$ and prescribed incoming data

$$f(x, \pm\pi, v) = p(x, \pm\pi, v) \quad v_z \lesseqgtr 0 \tag{2.11}$$

Due to the presence of the force G, the argument requires some care. We introduce $\kappa(z) = \mathrm{e}^{\varepsilon G(z+\pi)}$. Then we multiply (2.10) by $2fM\kappa$, integrate over $[-\pi,\pi]^2 \times \mathbb{R}^3$, and integrate by parts to get the Green identity

$$\| \kappa^{\frac12} \gamma^- f \|^2_{2,2,\sim} - \frac{1}{\varepsilon} (\kappa f, L_J f)_{2,2} = (\kappa g, f)_{2,2} + \| \kappa^{\frac12} p\|^2_{2,2,\sim}. \tag{2.12}$$

Apply the spectral inequality (2.7) to obtain, for small η

$$\| \kappa^{\frac12} \gamma^- f \|^2_{2,2,\sim} + \frac{c}{2\varepsilon} \| \kappa^{\frac12} \nu^{\frac12} (I - P_J) f \|^2_{2,2} \tag{2.13}$$

$$\leq c(\varepsilon \| \kappa^{\frac12} \nu^{-\frac12} (I - P_J) g \|^2_{2,2} + \eta \| \kappa^{\frac12} P_J f \|^2_{2,2} \tag{2.14}$$

$$+ \frac{1}{\eta} \| \kappa^{\frac12} P_J g \|^2_{2,2} + \| \kappa^{\frac12} p \|^2_{2,2,\sim}.$$

Finally, since $1 \leq \kappa(z) \leq \mathrm{e}^{2\varepsilon\pi G}$, we conclude that, for some constant C it results:

$$\| \gamma^- f \|^2_{2,2,\sim} + \frac{c}{2\varepsilon} \| \nu^{\frac12} (I - P_J) f \|^2_{2,2} \tag{2.15}$$

$$\leq C\Big(\varepsilon \| \nu^{-\frac12} (I - P_J) g \|^2_{2,2} + \eta \| P_J f \|^2_{2,2} + \frac{1}{\eta} \| P_J g \|^2_{2,2} + \| p \|^2_{2,2,\sim} \Big).$$

Inequality (2.16) and its variations will be referred to in the rest of the paper as Green inequality and used extensively, with L_J^* replacing sometimes L_J. In the last case the projector in the r.h.s. has to be replaced by $P_J^* = P$.

Lemma 2.1 *Let $\varphi(x,z,v)$ be solution to*

$$\mu v_x \frac{\partial \varphi}{\partial x} + v_z \frac{\partial \varphi}{\partial z} - \varepsilon G M^{-1} \frac{\partial (M\varphi)}{\partial v_z} = \frac{1}{\varepsilon} L_J^* \varphi + g, \tag{2.16}$$

periodic in x of period 2π, and with zero ingoing boundary values at $z = -\pi, \pi$. Then, for some small η

$$\| \nu^{\frac12} (I - P) \varphi \|_{2,2} \leq C\Big(\varepsilon \| \nu^{-\frac12} (I - P) g \|_{2,2} + \| P g \|_{2,2} + \eta\varepsilon \| < P\varphi > \|_2 \Big), \tag{2.17}$$

$$\| \widetilde{P\varphi} \|_{2,2} \leq C\Big(\| \nu^{-\frac12} (I - P) g \|_{2,2} + \frac{1}{\varepsilon} \| P g \|_{2,2} + \eta \| < P\varphi > \|_2 \Big). \tag{2.18}$$

Proof of Lemma 2.1. The method from [17] (a variant of [17] Scn 5.3) can be adapted to the present setting to obtain the existence of a solution to (2.16), if one includes the above spectral estimate for L_J^*, and the new characteristics curves due to the force term.

We write (2.16) in Fourier variables. For $\xi \in \mathbb{Z}^2 - \{(0,0)\}$,

$$i(\mu\ \xi_x v_x + \xi_z v_z)\hat{\varphi} = \frac{1}{\varepsilon}\widehat{L_J^*\varphi} + \varepsilon G M^{-1}\frac{\partial(M\hat{\varphi})}{\partial v_z} + \hat{g} - (-1)^{\xi_z}|v_z|r. \quad (2.19)$$

Here

$$r = \begin{cases} \mathcal{F}_x\varphi(\xi_x, \pi, v) & \text{for } v_z > 0, \\ \mathcal{F}_x\varphi(\xi_x, -\pi, v) & \text{for } v_z < 0. \end{cases} \quad (2.20)$$

Introduce

$$v^\mu = (\mu v_x, v_z), \quad \Phi = \hat{\varphi}, \quad \tilde{Z} = \varepsilon^{-1}\widehat{L_J^*\varphi} + \varepsilon G M^{-1}\frac{\partial(M\hat{\varphi})}{\partial v_z} + \hat{g} - (-1)^{\xi_z}|v_z|r,$$

$$Z = \varepsilon^{-1}\widehat{L_J^*\varphi} + \hat{g} - (-1)^{\xi_z}|v_z|r, \quad Z' = \varepsilon^{-1}\widehat{L_J^*\varphi} + \hat{g}, \quad \hat{U} = (i\xi\cdot v^\mu)^{-1}.$$

Let χ be the indicatrix function of the set $\{v \in \mathbb{R}^3 \mid \ \mid \xi\cdot v^\mu \mid < \alpha|\xi|\}$, for some positive α to be chosen later. Let $\zeta_s(v) = (1+ \mid v \mid)^s$. For $\xi \neq (0,0)$

$$\begin{aligned} \parallel P(\chi\Phi) \parallel \ &\leq c\sum_{j=0}^{4}\left|\int_{\mathbb{R}^3} dv\chi(v)\Phi(\xi,v)M(v)\psi_j(v)\right| \\ &\leq c \parallel \zeta_{-s}\chi\Phi \parallel \sum_{j=0}^{4} \parallel \chi\zeta_s\psi_j \parallel \leq c\sqrt{\alpha} \parallel \zeta_{-s}\chi\Phi \parallel . \end{aligned}$$

We use this estimate with the following choice of α, $\alpha = \parallel \zeta_{-s}\Phi \parallel^{-1} \parallel \zeta_{-s}Z' \parallel$.

We also introduce an indicatrix function χ_1 with $\alpha = \sqrt{\delta_1}$. We fix δ_1 so that $c\sqrt{\delta_1} << 1$. Then we find from the above estimate that the P-part of the right-hand side, $\parallel P(\chi_1\Phi) \parallel$, can be absorbed by $\parallel P(\chi_1\Phi) \parallel$ in the left-hand side. The estimates hold in the same way when χ_1 is suitably smoothed around $\sqrt{\delta_1}|\xi|$. For the remaining $\chi^c\chi_1^c\Phi = (1-\chi)(1-\chi_1)\Phi$, we shall use that $\Phi = -\hat{U}\tilde{Z}$. Then

$$\begin{aligned} \parallel P(\chi_1^c\chi^c\Phi) \parallel^2 \ &\leq c \parallel \zeta_{s+2}\chi_1^c\chi^c\hat{U} \parallel^2 \parallel \zeta_{-s}Z' \parallel^2 + \frac{c \parallel \sqrt{|v_z|}r \parallel^2}{\delta_1|\xi|^2} + \varepsilon G|\Theta| \\ &\leq \frac{c}{\mid \xi \mid^2 \mid \alpha \mid} \parallel \zeta_{-s}Z' \parallel^2 + \frac{c \parallel \sqrt{|v_z|}r \parallel^2}{\delta_1|\xi|^2} + \varepsilon G|\Theta|, \end{aligned}$$

with

$$\Theta = \sum_{j=0}^{4}\int \psi_j\chi_1^c\chi^c\hat{U}\frac{\partial(M\hat{\varphi})}{\partial v_z}dv\left(\int \psi_j\chi_1^c\chi^c(\hat{\varphi} - \hat{U}Z)Mdv\right)^* .$$

We replace α by $\parallel \zeta_{-s}\Phi \parallel^{-1} \parallel \zeta_{-s}Z' \parallel$ in the denominator. That gives

$$\begin{aligned}\parallel P\Phi \parallel^2 &\leq c(\parallel \zeta_{-s}\Phi \parallel \parallel \zeta_{-s}Z' \parallel + \frac{\parallel \sqrt{|v_z|}r \parallel^2}{\delta_1 |\xi|^2} \\ &+\delta_1 \parallel \zeta_{-s}(I-P)\Phi \parallel^2) + \varepsilon G|\Theta|.\end{aligned}$$

Hence,

$$\begin{aligned}\parallel P\Phi \parallel^2 &\leq c\Big((\parallel P\Phi \parallel + \parallel \zeta_{-s}(I-P)\Phi \parallel) \parallel \zeta_{-s}Z' \parallel + \frac{\parallel \sqrt{|v_z|}r \parallel^2}{\delta_1 |\xi|^2} \\ &+\delta_1 \parallel \zeta_{-s}(I-P)\Phi \parallel^2 \Big) + \varepsilon G|\Theta|.\end{aligned}$$

Consequently,

$$\begin{aligned}\parallel P\Phi \parallel^2 &\leq c\Big(\parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel \sqrt{|v_z|}r \parallel^2}{\delta_1 |\xi|^2} + \parallel \zeta_{-s}(I-P)\Phi \parallel \parallel \zeta_{-s}Z' \parallel \\ &+ \parallel \zeta_{-s}(I-P)\Phi \parallel^2 \Big) + \varepsilon G|\Theta|.\end{aligned}$$

We next discuss the term $\varepsilon G|\Theta|$. The first integral can be bounded by ε times an integral of a product of M, $1+|\xi_z|$, a polynomial in v, $|\hat{\varphi}|$ and $\hat{U}$ or $\hat{U}^2$. So this factor is bounded by $\varepsilon c \parallel \Phi \parallel$. And so,

$$\parallel P\Phi \parallel^2 \leq c\Big(\parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel \sqrt{|v_z|}r \parallel^2}{\delta_1 |\xi|^2} + \parallel (I-P)\Phi \parallel^2 \Big).$$

Therefore for $\xi \neq (0,0)$,

$$\begin{aligned}\int |P\Phi|^2(\xi,v)Mdv &\leq c\Big(\frac{1}{\varepsilon^2} \parallel \zeta_{-s}(v)\widehat{L_J^*\varphi}(\xi,\cdot) \parallel^2 + \parallel (I-P)\Phi(\xi,\cdot) \parallel^2 \\ &+ \frac{\parallel \sqrt{|v_z|}r \parallel^2}{\delta_1 |\xi|^2} + \parallel \nu^{-\frac{1}{2}}\hat{g}(\xi,\cdot) \parallel^2 \Big). \end{aligned} \tag{2.21}$$

We remember that the zero Fourier mode of $\widetilde{P\varphi}$ is zero by definition. Hence, taking ε small enough and summing over all $0 \neq \xi \in Z^2$, implies by Parseval and the spectral estimate on L_J^*, that

$$\begin{aligned}\int (\widetilde{P\varphi})^2(x,z,v)Mdvdxdz &\leq c\Big(\frac{1}{\varepsilon^2}\int \nu((I-P)\varphi)^2(x,z,v)Mdvdxdz \\ &+ \int \nu^{-1}g^2(x,z,v)Mdvdxdz + \parallel \gamma^-\varphi \parallel^2_{2,2,\sim} \Big) \end{aligned} \tag{2.22}$$

Inequality (2.16) implies (2.17). Replacing (2.16) in (2.22) gives (2.18), and this concludes the proof of the Lemma 2.1. □

Remark. The statement of Lemma 2.1 still holds if we replace the operator L_J^* with the operator L_J and the operator P with P_J, with some minor modification. The main change is due to the fact that one has to use the basis functions in $\texttt{Kern}(L_J)$ namely the $\bar{\psi}_j$'s, instead of the ψ_j's. They depend on (x,z), therefore we fix a point (x_0,z_0) and use the $\bar{\psi}_j$'s computed at this point and the corresponding projector P_{J0}. Then the argument of the proof can be repeated word by word. At the end, we replace P_{J0} with P_J since $P_J - P_{J0} = O(\varepsilon)$.

Put $H(R) = \sum_{n=1}^{5} \varepsilon^{n-1} J(\Phi_s^{(n)}, R)$ and decompose H in accordance with the operator L_J. Set $H_1(\,\cdot\,) = H(\,\cdot\,) - J(q, P\,\cdot)$. We notice that $H_1(\cdot)$ is of order zero in ε and only depends on the nonhydrodynamic projection $(I-P)$.

To pass from the linear results to the non linear case we will use an iteration procedure that will lead to Theorem 2.2 below. To separate the difficulties coming from the non linear term and from the boundary conditions, we split the remainder R in two parts, R_1 and R_2, solutions of two different equations. In the equation for R_1 the boundary conditions are of given indata type and the nonhydrodynamic known term is included, while in the equation for R_2 the boundary conditions are of diffusive type and the known term is absent. The equation for R_2 will be given later (see eq. (2.25)). We start with a discussion of the equation for R_1,

$$\mu\, v_x \frac{\partial R_1}{\partial x} + v_z \frac{\partial R_1}{\partial z} - \varepsilon G M^{-1} \frac{\partial (M R_1)}{\partial v_z} = \frac{1}{\varepsilon} L_J R_1 + H_1(R_1) + g, \tag{2.23}$$
$$R_1(x, \mp\pi, v) = -\frac{1}{\varepsilon} \bar{\psi}(x, \mp\pi, v), \quad v_z \gtrless 0.$$

Here R_1 is periodic in x of period 2π, and $L_J = L(\,\cdot\,) + \varepsilon J(q, P\,\cdot\,)$ has been introduced earlier. An existence proof for this problem can be obtained similarly to that for φ above.
The nonhydrodynamic part of R_1 is estimated similarly to the corresponding proof for Lemma 2.1,

$$\frac{1}{\varepsilon} \parallel \gamma^- R_1 \parallel_{2,2,\sim}^2 + \frac{c}{2\varepsilon^2} \parallel \nu^{\frac{1}{2}} (I-P_J) R_1 \parallel_{2,2}^2$$
$$\leq C\Big(\parallel \nu^{-\frac{1}{2}} (I-P_J) g \parallel_{2,2}^2 + \frac{\eta}{2\varepsilon} \parallel P_J R_1 \parallel_{2,2}^2 + \frac{1}{2\eta\varepsilon} \parallel P_J g \parallel_{2,2}^2$$
$$+ \frac{1}{\varepsilon^3} \parallel \bar{\psi} \parallel_{2,2,\sim}^2 \Big),$$

for small $\eta > 0$. Here we have used the fact that

$$|(R_1, H_1(R_1))_{2,2}| \leq C(\parallel \nu^{\frac{1}{2}} (I-P_J) R_1 \parallel_{2,2}^2 + \varepsilon^2 \parallel P_J R_1 \parallel_{2,2}^2).$$

A priori bounds for $P_J R_1$ will be based on dual techniques involving the problem (2.16). Consider first the problem for R_1 without the term $H_1(R_1)$. It holds

Lemma 2.2 *Set $h := P_J R_1$. Then there is $\delta_0 > 0$, such that for $0 < \delta < \delta_0$,*

$$\parallel h \parallel_{2,2}^2 \leq C(\parallel \nu^{-\frac{1}{2}} (I-P_J) g \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel P_J g \parallel_{2,2}^2 + \frac{1}{\varepsilon^3} \parallel \bar{\psi} \parallel_{2,2,\sim}^2).$$

Proof of Lemma 2.2. The function R_1 is 2π-periodic in x, and here solution to

$$\mu\ v_x\frac{\partial R_1}{\partial x}+v_z\frac{\partial R_1}{\partial z}-\varepsilon GM^{-1}\frac{\partial (MR_1)}{\partial v_z}=\frac{1}{\varepsilon}L_JR_1+g, \tag{2.24}$$
$$R_1(x,\mp\pi,v)=-\frac{1}{\varepsilon}\bar\psi(x,\mp\pi,v),\quad v_z\gtrless 0.$$

Let φ be a 2π-periodic function in x, solution to

$$\mu\ v_x\frac{\partial \varphi}{\partial x}+v_z\frac{\partial \varphi}{\partial z}-\varepsilon GM^{-1}\frac{\partial (M\varphi)}{\partial v_z}=\frac{1}{\varepsilon}L_J^*\varphi+h,$$

with zero ingoing boundary values at $z=-\pi,\pi$.

We consider the equation for R_1 multiplied by $M\kappa\varphi$ and the one for φ multiplied by $M\kappa R_1$ and add them. After integrating on $[-\pi,\pi]^2\times\mathbb{R}^3$ and integrating by parts, we obtain,

$$\int dxdzdvM\kappa\Big(\frac{\partial}{\partial z}(v_zR_1\varphi)-\varepsilon G\frac{\partial (R_1\varphi)}{\partial v_z}\Big)=\int dxdzdvM\kappa$$
$$\Big[\frac{1}{\varepsilon}((I-P)\varphi L_J(I-P_J)R_1)+\frac{1}{\varepsilon}(I-P_J)R_1L_{J^*}(I-P)\varphi+\kappa g\varphi+\kappa hR_1\Big].$$

Using again the bound $1\leq\kappa\leq e^{2\pi\varepsilon G}$ and the assumption $h=P_JR_1$, this gives

$$\parallel h\parallel_{2,2}^2\leq\frac{K_1}{2}\parallel\gamma^-R_1\parallel_{2,2,\sim}^2+\frac{1}{2K_1}\parallel\gamma^-\varphi\parallel_{2,2,\sim}^2+\frac{K_3}{\varepsilon}\parallel\nu^{\frac12}(I-P_J)R_1\parallel_{2,2}^2$$
$$+\frac{1}{K_3\varepsilon}\parallel\nu^{\frac12}(I-P)\varphi\parallel_{2,2}^2+\frac{K_4}{2}\parallel\nu^{-\frac12}(I-P_J)g\parallel_{2,2}^2$$
$$+\frac{1}{2K_4}\parallel\nu^{\frac12}(I-P)\varphi\parallel_{2,2}^2+\frac{\varepsilon^2}{2K_4}\parallel\nu^{\frac12}\varphi\parallel_{2,2}^2+\frac{K_2}{2}\parallel P_Jg\parallel_{2,2}^2$$
$$+\frac{1}{2K_2}\parallel P\varphi\parallel_{2,2}^2+\frac{\varepsilon^2}{2K_2}\parallel\nu^{\frac12}\varphi\parallel_{2,2}^2,$$

for arbitrary positive constants K_j, $j=1,...,4$. It then follows that

$$\parallel h\parallel_{2,2}^2\leq c[(\frac{K_1}{\varepsilon^2}+\frac{K_3}{\varepsilon^2})\parallel\bar\psi\parallel_{2,2,\sim}^2+(\varepsilon K_1+K_4+K_3\varepsilon)\parallel\nu^{-\frac12}(I-P_J)g\parallel_{2,2}^2$$
$$+(\frac{1}{\varepsilon K_1}+\frac{1}{\varepsilon^2K_2}+\frac{1}{\varepsilon K_3}+\frac{1}{K_4}+\frac{\eta_1}{\varepsilon^2K_1})\parallel h\parallel_{2,2}^2$$
$$+(\frac{K_1}{\varepsilon}+\frac{K_3}{\eta_1}+K_2)\parallel P_Jg\parallel_{2,2}^2$$
$$+(\eta_1K_1+\eta_1K_3)\parallel h\parallel_{2,2}^2+\eta(\frac{\eta_1}{K_1}+\frac{\varepsilon}{K_3}+\frac{\varepsilon^2}{K_4}+\frac{1}{K_2})\parallel<P\varphi>\parallel_2^2.$$

For an estimate of the final $<P\varphi>$-term when G is small and the Rayleigh number for the rolls lies in a neighbourhood of the bifurcation point, we may apply an exact, direct approach based on ordinary differential equations. Namely, $<\varphi>_x(\cdot):=\int\varphi(x,\cdot)dx$ satisfies a 1-d stationary laminar given indata problem, similar to eq. (3.5) in [1]. The present case is different from the one in [1] because there will be new contributions coming from the terms in the expansion depending on x, for example the term q in the J-term. Since the first order of the ε expansion is of order δ, the same is true for the higher order (in ε) terms. The new contributions can all be considered as deviations from their x-independent values at the bifurcation point, hence are of order δ. We include them in the estimate as $\eta \parallel \varphi \parallel_{2,2}$, η small, and obtain

$$\begin{aligned}\|< P\varphi >\|_2 &\leq c \,\|< P\varphi >_x\|_{2,2} \leq c(\|< P\varphi >_x\|_{2,2} + \varepsilon \parallel \varphi \parallel_{2,2}) \\ &\leq \frac{c}{\varepsilon} \|< h >_x\|_{2,2} + \eta \parallel \varphi \parallel_{2,2} \leq \frac{c}{\varepsilon} \parallel h \parallel_{2,2} + \eta \parallel \varphi \parallel_{2,2} .\end{aligned}$$

For details cf Lemma 3.5 in [1].
Choosing $\varepsilon << 1$, then K_1 and K_3 (resp.K_2) of order ε^{-1} (resp. ε^{-2}), η_1 of order ε, and using Lemma 2.1, the inequality of Lemma 2.2 follows. □

Remark. From here on, small factors η in the estimates will depend also on the small δ_0.

In the following lemma we get the final estimates for R_1.

Lemma 2.3 *If R_1 is a solution to the system* (2.23)*, then*

$$\begin{aligned}\parallel \nu^{\frac{1}{2}} R_1 \parallel_{2,2} &\leq c\Big(\parallel \nu^{-\frac{1}{2}}(I - P_J)g \parallel_{2,2} + \frac{1}{\varepsilon} \parallel P_J g \parallel_{2,2} + \varepsilon^{-\frac{3}{2}} \parallel \bar\psi \parallel_{2,2,\sim} \Big), \\ \parallel \nu^{\frac{1}{2}} R_1 \parallel_{\infty,2} &\leq c\Big(\frac{1}{\varepsilon} \parallel \nu^{-\frac{1}{2}}(I - P_J)g \parallel_{2,2} + \frac{1}{\varepsilon^2} \parallel P_J g \parallel_{2,2} + \varepsilon \parallel \nu^{-\frac{1}{2}} g \parallel_{\infty,2} \\ &\quad + \varepsilon^{-\frac{5}{2}} \parallel \bar\psi \parallel_{2,2,\sim} \Big).\end{aligned}$$

<u>Proof of Lemma 2.3.</u> Consider first the solution to (2.23) with $H_1 = 0$. It satisfies

$$\begin{aligned}&\parallel \gamma^- R_1 \parallel_{2,2,\sim} + \varepsilon^{-\frac{1}{2}} \parallel \nu^{\frac{1}{2}}(I - P_J)R_1 \parallel_{2,2} \leq \\ &c\Big(\frac{1}{\varepsilon} \parallel \bar\psi \parallel_{2,2,\sim} + \varepsilon^{\frac{1}{2}} \parallel \nu^{-\frac{1}{2}}(I - P_J)g \parallel_{2,2} + \eta_1 \parallel P_J R_1 \parallel_{2,2} + \frac{1}{\eta_1} \parallel P_J g \parallel_{2,2} \Big),\end{aligned}$$

for any $\eta_1 > 0$. By Lemma 2.4 and some additional computations using the solution formula,

$$\parallel \nu^{\frac{1}{2}} R_1 \parallel_{\infty,2} \leq c\Big(\frac{1}{\varepsilon} \parallel \nu^{\frac{1}{2}} R_1 \parallel_{2,2} + \varepsilon \parallel \nu^{-\frac{1}{2}} g \parallel_{\infty,2} + \parallel \gamma^+ R_1 \parallel_{2,2,\sim} \Big).$$

It is easy to see that adding the term $H_1(R_1)$ does not change the above results. That proves the lemma. □

Now we study R_2 , the other part of the remainder. Denote by

$$f^-(x, \mp\pi, v) = \frac{M_\mp(v)}{M(v)} \int_{w_z \lessgtr 0} \Big(R_1(x, \mp\pi, w) + \frac{1}{\varepsilon}\bar\psi(x, \mp\pi, w)\Big)|w_z| M dw, \; v_z \gtrless 0,$$

the incoming data for R_2 which is solution to

$$\mu v_x \frac{\partial R_2}{\partial x} + v_z \frac{\partial R_2}{\partial z} - \varepsilon G M^{-1} \frac{\partial (M R_2)}{\partial v_z} = \frac{1}{\varepsilon} L_J R_2 + H_1(R_2), \tag{2.25}$$

$$R_2(x, \mp\pi, v) = f^-(x, \mp\pi, v) + \frac{M_\mp(v)}{M(v)} \int_{w_z \lesseqgtr 0} R_2(x, \mp\pi, w)|w_z| M dw, \quad v_z \gtreqless 0. \tag{2.26}$$

Existence and uniqueness for (2.25) follow as in the laminar case, cf [1]. The following a priori estimates hold for R_2. By Green's formula and the definition of $H_1(R_2) = H_1((I-P)R_2)$,

$$\| \gamma^- R_2 \|^2_{2,2,\sim} + \frac{c}{\varepsilon} \| \nu^{\frac{1}{2}} (I - P_J) R_2 \|^2_{2,2} \le \| \gamma^+ R_2 \|^2_{2,2,\sim} .$$

Here we face a problem of diffusive boundary conditions. The ingoing flow is given for φ and for R_1 but not for R_2. The following bound is proved as in equation (4.23) in [9],

$$\mp \int_{\mathbb{R}^3} v_z R_2^2(x, \pm\pi, v) dv \le c\varepsilon\eta \int_{v_z \gtreqless 0} |v_z| M R_2^2(x, \pm\pi, v) dv + \frac{1}{\varepsilon\eta} \int_{v_z \lesseqgtr 0} |v_z| M (f^-(x, \pm\pi, v))^2 dv.$$

The computation in [9] has to be adapted to the fact that the boundary conditions for R_2 are not purely diffusive but contain the given data f^-. From Green's formula we have also

$$\frac{c_1}{\varepsilon} \| \nu^{\frac{1}{2}} (I - P_J) R_2 \|^2_{2,2} \le \int \big((v_z, R_2^2(x, -\pi, v)) - (v_z, R_2^2(x, \pi, v)) \big) dx.$$

Then, using the previous bound we get for any $\eta > 0$,

$$\frac{c_1}{\varepsilon} \| \nu^{\frac{1}{2}} (I - P_J) R_2 \|^2_{2,2} \le c\varepsilon\eta \sum_{\pm} \int_{v_z \gtreqless 0} |v_z| M R_2^2(x, \pm\pi, v) dx dv + \frac{1}{\varepsilon\eta} \| f^- \|^2_{2,2,\sim} . \tag{2.27}$$

We need to estimate the terms involving the outgoing parts of R_2 in the r.h.s. in terms of $\|R_2\|_{2,2}$ and $\|f^-\|_{2,2,\sim}$. This is done separately at $\pm\pi$. We start with π. Consider the equation for R_2, multiply by $\kappa M R_2$ and integrate in velocity over the region $v_z \ge q$. Then integrate over space, using a smooth cut-off function $\chi(z)$ which is 0 in a small interval close to $-\pi$ and 1 close to π. Finally, integrate over q, for $q_0 < q \le 0$ and q_0 small enough. We get

$$\int_{q_0}^0 dq \int_{v_z \ge q} dv \int_{-\pi}^{\pi} dx v_z \kappa(\pi) M R_2^2(x, \pi, v) \le \frac{c_1 q_0}{\varepsilon^2} \| \nu^{\frac{1}{2}} (I - P_J) R_2 \|^2_{2,2}$$
$$- \int_{q_0}^0 dq \int_{v_z \ge q} dv \int_{[-\pi,\pi]^2} dx dz \kappa(z) \chi'(z) v_z M R_2^2 + c\varepsilon G \| R_2 \|^2_{2,2} .$$

The term on the l.h.s. equals

$$C|q_0| \parallel \gamma^- R_2 \parallel^2_{2,2\sim} + \int_{q_0}^0 dq \int_{q<v_z<0} dv \int_{-\pi}^{\pi} dx v_z M R_2^2(x,\pi,v).$$

Here in the last term we can replace R_2 by the ingoing boundary data so to estimate it as

$$\Big| \int_{q_0}^0 dq \int_{q<v_z<0} dv \int_{-\pi}^{\pi} dx \big[M^+ \int_{w_z\geq 0} dw \int_{-\pi}^{\pi} dx w_z M R_2(x,\pi,w) + M f^-(\pi)\big]^2 \Big|$$
$$\leq c(q_0)\big[\parallel \gamma^- R_2 \parallel^2_{2,2\sim} + \parallel f^- \parallel^2_{2,2,\sim} \big],$$

with $c(q_0) = o(|q_0|)$. Adding a similar estimate at $-\pi$ gives

$$\parallel \gamma^- R_2 \parallel^2_{2,2\sim} \leq \frac{C}{\varepsilon^2} \parallel \nu^{\frac12}(I-P_J)R_2 \parallel^2_{2,2} + C \parallel f^- \parallel^2_{2,2,\sim} + C \parallel P_J R_2 \parallel^2_{2,2} . \tag{2.28}$$

Replacing in (2.27) we get

$$\frac{1}{\varepsilon} \parallel \nu^{\frac12}(I-P_J)R_2 \parallel^2_{2,2} \leq \varepsilon\eta C \parallel P_J R_2 \parallel^2_{2,2} + \frac{c}{\varepsilon\eta} \parallel f^- \parallel^2_{2,2,\sim} . \tag{2.29}$$

We shall next prove an a priori estimate for the hydrodynamic part of R_2. First we consider the 1-d (x-independent) case and then include in the argument the missing terms which, as explained before, will be of order δ. The extension of the 1-d results to the 2-d case will be based on perturbative arguments in δ. To take into account these terms of order δ we add to the right-hand side an inhomogeneous term g_1 with $\int M g_1 dv = 0$, which will be of use later on in the proof of Lemma 2.5.

Lemma 2.4 *Let $R_2(z,v)$ be solution of the 1-d problem and f^- be defined as before,*

$$v_z \frac{\partial R_2}{\partial z} - \varepsilon G M^{-1} \frac{\partial (M R_2)}{\partial v_z} = \frac{1}{\varepsilon} L_J R_2 + H_1(R_2) + g_1, \tag{2.30}$$
$$R_2(\mp\pi, v) = \frac{M_\mp(v)}{M(v)} \int_{w_z \lesseqgtr 0} R_2(\mp\pi, w)|w_z| M dw + f^-(\mp\pi, v), \quad v_z \gtrless 0.$$

If $g_{10} = \int_{\mathbb{R}^3} dv M g_1 = 0$, then it holds that

$$\parallel P_J R_2 \parallel^2_{2,2} \leq \frac{c}{\varepsilon^2} \parallel f^- \parallel^2_{2\sim} + \parallel \nu^{-\frac12} g_1 \parallel^2_2 .$$

Proof of Lemma 2.4. Here, L_J is generated by a function $\tilde q$, independent of x, defined by $\tilde q = q + O(\delta)$. In the same way, in the term H_1 we retain only terms of order zero in δ, which are independent of x, and include the remaining terms in g_1. In this way we reduce the equation to a 1-d equation with a given term.

Consider equation (2.30) for $\hat{R}_2 = \mathcal{F}_z R_2$, the Fourier-transform in z of R_2. It satisfies the equation

$$iv_z\xi_z\hat{R}_2 - \varepsilon GM^{-1}\frac{\partial}{\partial v_z}(M\hat{R}_2) = \varepsilon^{-1}\widehat{L_JR_2} + \widehat{H_1(R_2)} - v_z r(-1)^{\xi_z} + \hat{g}_1 \quad (2.31)$$

with $r(v)$ now denoting the difference between ingoing and outgoing boundary values,

$$r(v) = R_2(\pi, v) - R_2(-\pi, v). \quad (2.32)$$

For $\xi_z \neq 0$ we use the method of Lemma 2.1 with L_J instead of its adjoint. Consider the ingoing boundary values as known, and follow step by step the proof of Lemma 2.1 with obvious changes. Let L_{J0} be the operator defined in (2.5) with q taken at a fixed (x_0, z_0). Define P_{J0} as the orthogonal projection on $\mathtt{Kern}(L_{J0})$. We reach the analogue of (2.21),

$$\begin{aligned}\int |P_{J0}\hat{R}_2(\xi_z, v)|^2 M dv \leq C\Big(&\frac{1}{\varepsilon^2} \parallel \zeta_{-s}(v)\widehat{L_JR_2}(\xi_z, \cdot) \parallel^2\\ &+ \parallel (I - P_{J0})\hat{R}_2(\xi_z, \cdot) \parallel^2 + \frac{\parallel P_{J0}(\chi_1^c\chi^c v_z r) \parallel^2}{\delta_1|\xi_z|^2}\\ &+ \parallel \nu^{-\frac{1}{2}}\hat{g}_1(\xi_z, \cdot) \parallel^2 + \parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_z, \cdot) \parallel^2 \Big).\end{aligned} \quad (2.33)$$

For an estimate of the r-term, we express it with the $\xi_z = 0$ term in the Fourier series for (2.31),

$$v_z r(v) = \frac{1}{\varepsilon}\widehat{L_JR_2}(0, v) + \varepsilon GM^{-1}\frac{\partial}{\partial v_z}(M\hat{R}_2(0, v)) + \hat{g_1}(0, v) + \widehat{H_1(R_2)}(0, v). \quad (2.34)$$

Inserting this into (2.33), and summing over $\xi_z \neq 0$, results in

$$\begin{aligned}\int (\widetilde{P_{J0}R_2})^2(z, v) M dv dz \ \leq C\Big(&\frac{1}{\varepsilon^2}\int \nu((I - P_J)R_2)^2(z, v) M dv dz\\ &+ \int \nu^{-1} g_1^2(x, z, v) M dv dz + \varepsilon^2 \parallel R_2 \parallel_{2,2}^2 \Big).\end{aligned}$$

We are left with the Fourier component $P_{J0}\hat{R}_2(\xi_z)$ for $\xi_z = 0$. Estimate separately the $(I - P)$-component and the P-component of $P_{J0}\hat{R}_2(0, \cdot)$. For $(I - P)P_{J0}\hat{R}_2(0, \cdot)$ we obtain

$$\parallel (I - P)P_{J0}\hat{R}_2(0, \cdot) \parallel \leq C\varepsilon(\parallel (I - P_J)R_2 \parallel_{2,2} + \parallel P_J R_2 \parallel_{2,2}).$$

The P-moments are the more involved and will be discussed each separately. We start from the v_z-moment of $\hat{R}_2(0, v)$. Multiply (2.30) by M and integrate over $z \in [-\pi, z]$ and v. Since $g_{10} = 0$, we have

$$\int v_z M R_2(z, v) dv \ = \int_{v_z > 0} f^-(-\pi, v) v_z M dv \ .$$

Given two functions $h(v)$ and $f(\,\cdot\,,v)$ we use the notation $f_h(\,\cdot\,) := \int dv h(v) f(\cdot,v)$. In particular, for $h=\psi_j$, $j=0,\ldots,4$, we also use the notation f_j. We have

$$|\hat{R}_{2v_z}(0)| = \left|\int v_z M R_2(z,v) dv dz\right| \leq C \parallel f^- \parallel_{2\sim} .$$

To estimate the moments of $\hat{R}_2(0,v)$ we use the identity

$$\hat{R}_2(0,v) = \varDelta - \sum_{\xi_z \neq 0} \hat{R}_2(\xi_z,v)(-1)^{\xi_z}, \tag{2.35}$$

where $\varDelta(v) = \pi(R_2(\pi,v) + R_2(-\pi,v))$, which follows from

$$R_2(\pi,v) + R_2(-\pi,v) = \frac{1}{2\pi}\sum_{\xi_z\in\mathbb{Z}} \hat{R}_2(\xi_z,v)(e^{i\pi\xi_z} + e^{-i\pi\xi_z}),$$

by solving for the $\xi_z = 0$ coefficient.

We write $\varDelta = 2\pi R_2(-\pi,v) + \pi r(v)$ for $v_z > 0$ and $\varDelta = 2\pi R_2(\pi,v) - \pi r(v)$ for $v_z < 0$.

We consider first the ψ_4-moment of $\hat{R}_2(0,v)$, denoted by $\hat{R}_{24}$. We notice that

$$\hat{R}_{2v_z^2\bar{A}}(0) = \frac{1}{\sqrt{6}}\hat{R}_{24}(0)\int v_z^2 v^2 \bar{A} M dv + \hat{R}^{\perp}_{2v_z^2\bar{A}}(0), \tag{2.36}$$

where $\hat{R}_2^{\perp} = (1-P)\hat{R}_2$ and $\bar{A}$ and $\bar{B}$ are nonhydrodynamic solutions to

$$L(v_z\bar{A}) = v_z(v^2 - 5T), \quad L(v_x v_z \bar{B}) = v_x v_z. \tag{2.37}$$

Hence, we are left with the control of $\hat{R}_{2v_z^2\bar{A}}(0)$. For that, we use (2.35). A multiplication of (2.35) with $Mv_z^2\bar{A}$ followed by a v-integration, gives

$$\hat{R}_{2v_z^2\bar{A}}(0) = \varDelta_{v_z^2\bar{A}} - \sum_{\xi\neq 0}\hat{R}_{2v_z^2\bar{A}}(\xi_z)(-1)^{\xi_z}.$$

Let us first consider the contribution $\varDelta_{v_z^2\bar{A}}$. Using the relation

$$\varDelta = 2\pi R_2(\pm\pi,v) \mp \pi r(v) \text{ for } v_z \lessgtr 0,$$

we notice that the first part it is computed in terms of the outgoing flow and then can be bounded in terms of $\int dv v_z^2 \bar{A} M f^-$ and $\int_{v_z \gtrless 0} v_z^2 \bar{A} R_2(\pm\pi,v) M dv$. Then, we multiply (2.30) by $M\chi v_z \bar{A}$ and integrate over $v_z > 0$ (similarly at $-\pi$) , to get the bound

$$\begin{aligned} \Big| \int_{v_z \gtrless 0} v_z^2 \bar{A} R_2(\pm\pi,v) M dv \Big|^2 \ &\leq \eta \parallel P_J R_2 \parallel_{2,2}^2 + c(\frac{1}{\varepsilon^2} \parallel (I-P_J)R_2 \parallel_{2,2}^2 \\ &+ \parallel \nu^{-\frac{1}{2}} g_{1,\perp} \parallel_{2,2}^2) \ . \end{aligned}$$

The second part, namely the $v_z^2\bar{A}$-moment of the πr-term, is estimated as before using (2.34).

In order to control $\hat{R}_{2v_z^2\bar{A}}(\xi)$ for $\xi_z \neq 0$ we take the inner product of (2.31) with $v_z\bar{A}$,

$$(-1)^{\xi_z} r_{v_z^2\bar{A}} \; + i\xi_z(v_z^2\bar{A}, \hat{R}_2(\xi_z)) + \varepsilon G \int \frac{\partial}{\partial v_z}(v_z\bar{A}) M\hat{R}_2(\xi_z)dv$$
$$= \frac{1}{\varepsilon}(v_z\bar{A}, \widehat{L_J R_2}(\xi_z)) + \hat{g}_{1,v_z\bar{A}}(\xi_z) + (v_z\bar{A}, \widehat{H_1(R_2)}(\xi_z)).$$

Hence

$$\hat{R}_{2v_z^2\bar{A}}(\xi_z) \; = \frac{(-1)^{\xi_z}}{\xi_z} i r_{v_z^2\bar{A}} + \frac{i\varepsilon G}{\xi_z}\int \frac{\partial}{\partial v_z}(v_z\bar{A}) M\hat{R}_2(\xi_z)dv$$
$$- \frac{i}{\varepsilon\xi_z}(v_z\bar{A}, \widehat{L_J R_2}(\xi_z)) - \frac{i}{\xi_z}\hat{g}_{1,v_z\bar{A}}(\xi_z) - \frac{i}{\xi_z}(v_z\bar{A}, \widehat{H_1(R_2)}(\xi_z)).$$

Combining this with (2.35) and noticing the pairwise cancellation of the r-terms with positive and negative ξ's, gives

$$|\hat{R}_{2v_z^2\bar{A}}(0)|^2 \leq c(\frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}} g_{1,\perp} \parallel_{2,2}^2 + \eta \parallel P_J R_2 \parallel_{2,2}^2),$$

and so using (2.29) and (2.36)

$$|\hat{R}_{24}(0)|^2 \; \leq c(\frac{1}{\varepsilon^2} \parallel \nu^{-\frac{1}{2}}(I-P_J)R_2 \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}} g_{1,\perp} \parallel_{2,2}^2) + \eta \parallel P_J R_2 \parallel_{2,2}^2$$
$$\leq c(\frac{1}{\varepsilon^2} \parallel f^- \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}} g_{1,\perp} \parallel_{2,2}^2) + \eta \parallel P_J R_2 \parallel_{2,2}^2 .$$

The v_x- and v_y- moments are analogous to each others, so we only discuss the former. This can be treated similarly to the previous $\hat{R}_{24}(0)$ case but with $\Delta = 2\pi R_2(\mp\pi, v) \pm \pi r(v)$, $v_z \lesseqgtr 0$. The $2\pi R_2$-term is now ingoing, and its inner product with v_x gives zero. The πr-term is estimated as before. For the sum in (2.35) of the other Fourier coefficients, we notice that $\hat{R}_{2v_x}(\xi) = \hat{R}_{2v_z^2 v_x}(\xi) - \hat{R}_{2\perp v_z^2 v_x}(\xi)$, and we can proceed as before. Since $PP_J\hat{R}_2$ differs from $P\hat{R}_2$ by terms of order ε which are already under control, we can summarize the results so far as

$$\int \Big(|\hat{R}_{2\bar{\psi}_1}(0)|^2 + |\hat{R}_{2\bar{\psi}_2}(0)|^2 + |\hat{R}_{2\bar{\psi}_3}(0)|^2 + |\hat{R}_{2\bar{\psi}_4}(0)|^2\Big) dz \qquad (2.38)$$
$$\leq c(\frac{1}{\varepsilon^2} \parallel f^- \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}} g_{1,\perp} \parallel_{2,2}^2) + \eta \parallel \hat{R}_2 \parallel_{2,2}^2 .$$

Finally for the $\hat{R}_{20}(0)$-moment, start by considering (2.30) with the new boundary conditions

$$R_2(-\pi, v) = f^-(-\pi, v), \quad v_z > 0, \qquad (2.39)$$
$$R_2(\pi, v) = \frac{M_+(v)}{M(v)} \int_{w_z>0} \big[R_2(\pi, w) - f^-(-\pi, w)\big] w_z M(w)dw, \; v_z < 0.$$

The new boundary values are constructed in such a way that $\int_{v_z<0} R_2(-\pi,v)v_zMdv = 0$ and this property will allow the new boundary conditions to be equivalent to the old ones. In fact, since $\int_{v_z<0} M_+(v)v_zdv = -1$,

$$\int_{\mathbb{R}^3} v_zR_2(\pi,v)M(v)dv = \int_{v_z<0} v_zR_2(\pi,v)M(v)dv + \int_{v_z>0} v_zR_2(\pi,v)M(v)dv$$
$$= -\int_{w_z>0} w_z[R_2(\pi,w) - f^-(-\pi,w)]M(w)dw + \int_{v_z>0} v_zR_2(\pi,v)M(v)dv$$
$$= \int_{w_z>0} w_zf^-(-\pi,v)M(v)dv.$$

Existence and uniqueness for the new problem are well known. We shall verify that the new problem also satisfies the old boundary conditions of (2.30) . At $z=-\pi$, for $v_z>0$ the new ingoing boundary condition for R_2 is $R_2(-\pi,v) = f^-(-\pi,v))$, which coincides with the second equation in (2.30) since $\sqrt{2\pi}\int_{w_z<0} R_2(-\pi,w)|w_z|Mdw = 0$. At $z=\pi$ the new ingoing boundary condition is

$$R_2(\pi,v) = M^{-1}M_+\int_{w_z>0}\big[R_2(\pi,w) - f^-(-\pi,w)\big]w_zM(w)dw, \quad v_z<0\ .$$

This coincides with the old boundary condition at π, given by the second equation in (2.30), that is with

$$R_2(\pi,v) = f^-(\pi,v) + M^{-1}M_+\int_{w_z>0} R_2(\pi,w)M(w)w_zdw, \quad v_z<0,$$

provided that

$$-f^-(\pi,v) = M^{-1}M_+\int_{w_z>0} f^-(-\pi,w)w_zM(w)dw, \quad v_z<0,$$

or, recalling the definition of f^-,

$$\int_{w_z>0}\big(R_1(\pi,w) + \frac{\bar\psi(\pi,w)}{\varepsilon}\big)M(w)w_zdw = \int_{w_z<0}\big(R_1(-\pi,w) + \frac{\bar\psi(-\pi,w)}{\varepsilon}\big)M(w)w_zdw. \quad (2.40)$$

To check this we note that from the assumption that the inhomogeneous term g in the equation for R_1 is such that $\int_{\mathbb{R}^3} dvM(v)g(v) = 0$, it follows that

$$(v_z, R_1(\pi,v)) = (v_z, R_1(-\pi,v)).$$

Then, using the boundary conditions for R_1 this becomes

$$\int_{v_z>0} M(v)R_1(\pi,v)v_zdv - \int_{v_z<0} M(v)\frac{\bar\psi}{\varepsilon}(\pi,v)v_zdv$$
$$+\int_{v_z>0} M(v)\frac{\bar\psi}{\varepsilon}(\pi,v)(-\pi,v)v_zdv - \int_{v_z<0} M(v)R_1(-\pi,v)v_zdv = 0.$$

Using $(\bar{\psi}(\pm\pi, v), v_z) = 0$, the claimed equivalence follows. Thus R_2 with the new boundary conditions equals the unique solution to (2.30).

We write (2.30) with $R_2' = R_2\kappa(z)$. The left- hand side becomes

$$\kappa^{-1}(z)\left(v_z\frac{\partial R_2'}{\partial z} - \varepsilon G\frac{\partial R_2'}{\partial v_z}\right).$$

Multiply the equation for R_2' by $Mv_z\kappa(z)$ and integrate over $[-\pi, z]\times\mathbb{R}^3$. The l.h.s. gives

$$\int R_2'(z,v)Mv_z^2dv - \int R_2'(-\pi,v)Mv_z^2dv$$
$$-\varepsilon G\int_{-\pi}^{z}dz'\int R_2'(z',v)Mv_z^2dv + \varepsilon G\int_{-\pi}^{z}dz'\int R_2'(z',v)Mdv\ .$$

Since $\int dvv_z^2R_2Mdv = R_{20} + \frac{2R_{24}}{\sqrt{6}} + \int dvv_z^2R_2^{\perp}Mdv$, by integrating the equation again over the interval $[-\pi,\pi]$, we control $|\hat{R}_{20}(0)|$ in terms of known quantities plus terms bounded in the $\|\cdot\|_{2,2}$-norm, multiplied by a factor ε, and the integral $\int R_2'(-\pi,v)Mv_z^2dv$. The contribution due to the incoming part is given by f^-. Therefore, we need an estimate of the outgoing boundary term $\int_{v_z<0} R_2(-\pi,v)Mv_z^2dv$. To do this, we repeat the steps above but this time integrate over $\{[-\pi,\pi]\times\mathbb{R}^3; v_z>0\}$, to get on the l.h.s.

$$\int_{v_z>0} R_2'(\pi,v)Mv_z^2dv - \int_{v_z>0} R_2'(-\pi,v)Mv_z^2dv$$
$$-\varepsilon G\int_{-\pi}^{\pi}dz'\int_{v_z>0} R_2'(z',v)Mv_z^2dv + \varepsilon G\int_{-\pi}^{\pi}dz'\int_{v_z>0} R_2'(z',v)Mdv.$$

Since R_2' solves the problem with modified boundary conditions, the incoming part in $-\pi$ can be bounded by $\|f^-\|_{2,2,\sim}$. Therefore we get an estimate of $\int_{v_z>0} R_2(\pi,v)Mv_z^2dv$ in terms of the norm of f^- and again quantities bounded in the $\|\cdot\|_{2,2}$-norm multiplied by a factor ε. To obtain the estimate of $\int_{v_z<0} R_2(-\pi,v)Mv_z^2dv$, we integrate over $[-\pi,\pi]\times\mathbb{R}^3$. The final estimate is

$$|\hat{R}_{20}(0)| \ \le c(\frac{1}{\varepsilon}\parallel f^-\parallel_{2\sim} + \frac{1}{\varepsilon}\parallel \nu^{\frac12}(I-P_J)R_2\parallel_{2,2}$$
$$+\eta\parallel R_2\parallel_{2,2} + \parallel \nu^{-\frac12}g_1\parallel_{2,2}).$$

By combining with the other moment estimates (2.38), and using (2.29), we obtain that

$$\parallel P_JR_2\parallel_{2,2}^2 \le \frac{c}{\varepsilon^2}\parallel f^-\parallel_{2\sim}^2 + \parallel \nu^{-\frac12}g_1\parallel_{2,2}^2\ . \tag{2.41}$$

This completes the proof of the lemma. □

Based on this 1-d analysis, it follows in the 2-d case that

Lemma 2.5 *The solution R_2 to (2.30) satisfies*

$$\| P_J R_2 \|_{2,2}^2 \leq c\frac{1}{\varepsilon^2} \| f^- \|_{2\sim}^2 .$$

Proof of Lemma 2.5. Consider equation (2.30) for the Fourier transform in x, z of R_2, $\hat{R}_2 = \mathcal{F}_x\mathcal{F}_z R_2$,

$$\begin{aligned} & i(\mu\xi_x v_x + \xi_z v_z)\hat{R}_2 \\ & = \frac{1}{\varepsilon}\widehat{L_J R_2} + \varepsilon G M^{-1}\frac{\partial(M\hat{R}_2)}{\partial v_z} + \widehat{H_1(R_2)} - v_z r(\xi_x, v)(-1)^{\xi_z}, \quad (2.42) \\ & r(\xi_x, v) = \mathcal{F}_x R_2(\xi_x, \pi, v) - \mathcal{F}_x R_2(\xi_x, -\pi, v). \quad (2.43) \end{aligned}$$

In the case $\xi_x \neq 0$, $\xi_z \neq 0$ we can reach, as in the case of Lemma 2.4, a bound like (2.33). If we consider the Fourier components with ξ large, we see that in (2.33) the r-terms are multiplied by a small number for ξ large and then can be estimated, by using (2.28), by $\eta \| R_2 \|_{2,2}$, with a small η, plus the earlier terms. We notice that in the case $\xi_x = 0$, equation (2.42) reduces to (2.31). Hence, we can apply Lemma 2.4 to $\hat{R}_2(0, \xi_z, v) = \int R_2(x, z, v)dx$ and, taking into account that g_1 is of order δ, get a bound for the Fourier components $P_J\hat{R}_2(0, \xi_z)$, for δ small. The remaining components in the case $\xi_x \neq 0$, ξ_x, ξ_z bounded, can be estimated in the following way.
We start from the moment $\hat{R}_{2v_x}$. Notice that $r_{v_z}(\xi_x) = \hat{f}^-_{v_z}(\xi_x)$, where $\hat{f}^-$ is the function defined as

$$\hat{f}^-(\xi_x, v) = \begin{cases} (\mathcal{F}_x f^-)(\xi_x, \pi, v) & \text{for } v_z < 0, \\ (\mathcal{F}_x f^-)(\xi_x, -\pi, v) & \text{for } v_z > 0. \end{cases}$$

Hence, by integrating (2.42) we obtain

$$|\xi_x||\hat{R}_{2v_x}(\xi_x, 0)| \leq C|\hat{f}^-_{v_z}(\xi_x)|.$$

Then, we look for a bound for $\hat{R}_{2v_z}(\xi_x, \xi_z)$ when $\xi_z \neq 0$. We use the function $\bar{B}$ introduced in (2.37), $\bar{B}$ being the solution of the equation $L(v_x v_z \bar{B}) = v_x v_z$.

We have $(v_z^2 v_x \bar{B}, \hat{R}_2) = (v_z^2 v_x \bar{B}, (I-P)\hat{R}_2) + (\psi_x, v_x v_z^2 \bar{B})(\psi_x, \hat{R}_2)$, where ψ_j, $j = 0, \ldots, 4$ are as usual the vectors of the orthonormal basis in $\mathtt{Kern}(L)$. We multiply (2.42), written for $\xi_z = 0$, by $v_z^2\bar{B} - (\psi_x, v_x v_z^2 \bar{B})$ and integrate over velocities. We first use the relation so obtained for $\xi_z = 0$ and obtain,

$$\begin{aligned} |r_{v_z^3\bar{B}}(\xi_x)| \ & \leq c(\frac{1}{\varepsilon} \| \nu^{-\frac{1}{2}} L_J(\widehat{I - P_J})R_2(\xi_x, 0) \|_2 + \varepsilon G \| \hat{R}_2(\xi_x, 0) \|_2 \\ & + |\hat{f}^-_{v_z}(\xi_x)| + \| \nu^{-\frac{1}{2}} \widehat{H_1(R_2)}(\xi_x, 0) \|_2 + \| (I-P)\hat{R}_2(\xi_x, 0) \|_2), \end{aligned}$$

having used the fact that $|r_{v_x}| \leq c|f^-_{v_x}|$. We use again that relation for $\xi_z \neq 0$. Since $(v_z^3\bar{B}, R_2) = (v_z^3\bar{B}, (I-P)R_2) + (R_2, \psi_z)(\psi_z, v_z^3\bar{B})$, we obtain in this way an expression for $\xi_z\hat{R}_{2v_z}(\xi_x, \xi_z)$, for $\xi_z \neq 0$, in terms of quantities under control, since with the previous subtraction we have removed from the equation the term $(\psi_x, \hat{R}_2)$. As a result, for $\xi_z \neq 0$,

$$\begin{aligned}|\xi_z|\mid \hat{R}_{2v_z}(\xi_x,\xi_z)\mid\ &\leq C\Big(\frac{1}{\varepsilon}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,\xi_z)\parallel\\&+\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,\xi_z)\parallel+\varepsilon\parallel \hat{R}_2(\xi_x,\xi_z)\parallel\\&+\frac{1}{\varepsilon}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,0)\parallel+\varepsilon\parallel \hat{R}_2(\xi_x,0)\parallel\\&+\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,0)\parallel+|\xi_x|\parallel \hat{f}^-(\xi_x)\parallel\Big).\end{aligned}$$

Notice that the last term is bounded because ξ_x is bounded in this part of the proof, the terms with ξ large having been estimated before.

In the same way the v_x-moment can be controlled when $\xi_z \neq 0$,

$$\begin{aligned}|\xi_x|\mid \hat{R}_{2v_x}(\xi_x,\xi_z)\mid&\leq C(\frac{1}{\varepsilon}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,\xi_z)\parallel\\&+\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,\xi_z)\parallel+\varepsilon\parallel \hat{R}(\xi_x,\xi_z)\parallel\\&+\frac{1}{\varepsilon}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,0)\parallel_2+\varepsilon\parallel \hat{R}_2(\xi_x,0)\parallel\\&+\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,0)\parallel+|f^-_{v_z}(\xi_x)|).\end{aligned}$$

For the v_z-moment $\hat{R}_{2v_z}(\xi_x,0)$, by using the proof of Lemma 2.4,

$$|\hat{R}_{2v_z}(\xi_x,0)|\leq c(\parallel \hat{f}^-(\xi_x)\parallel+\parallel \mathcal{F}_xR_{2v_x}(\xi_x)\parallel).$$

To proceed, we observe that

$$\begin{aligned}\parallel \mathcal{F}_xR_{2v_x}(\xi_x)\parallel\ &\leq C\Big(\frac{1}{\varepsilon}(\sum_{\xi_z}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,\xi_z)\parallel^2)^{\frac{1}{2}}\\&+(\sum_{\xi_z}\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,\xi_z)\parallel^2)^{\frac{1}{2}}+\varepsilon(\sum_{\xi_z}\parallel \hat{R}_2(\xi_x,\xi_z)\parallel^2)^{\frac{1}{2}}\\&+\frac{1}{\varepsilon}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,0)\parallel+\varepsilon\parallel \hat{R}_2(\xi_x,0)\parallel+|\hat{f}^-_{v_z}(\xi_x)|\\&+\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,0)\parallel\Big).\end{aligned}$$

And so,

$$\begin{aligned}|\hat{R}_{2v_z}(\xi_x,0)|\ &\leq C\Big(\parallel \hat{f}^-(\xi_x)\parallel+\varepsilon\parallel R_2\parallel_{2,2}\\&+\frac{1}{\varepsilon}(\sum_{\xi_z}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,\xi_z)\parallel^2)^{\frac{1}{2}}\\&+(\sum_{\xi_z}\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,\xi_z)\parallel^2)^{\frac{1}{2}}+\frac{1}{\varepsilon}\parallel \nu^{-\frac{1}{2}}L_J\widehat{(I-P_J)}R_2(\xi_x,0)\parallel\\&+\parallel \nu^{-\frac{1}{2}}\widehat{H_1(R_2)}(\xi_x,0)\parallel\Big).\end{aligned}$$

To estimate $\hat{R}_{2v_y}(\xi_x, 0)$, we use the method of Lemma 2.4, that is the proof of (2.38). Then, to get the estimate for $\xi_z \neq 0$, we multiply (2.42) by $v_y v_z$ and use $v_x v_y v_z \in (\mathtt{Kern}(L))^\perp$ to estimate the term in r as

$$\begin{aligned}|r_{v_z^2 v_y}(\xi_x)| &\leq C(\frac{1}{\varepsilon} \parallel \nu^{-\frac{1}{2}} L_J(\widehat{I-P_J}) R_2(\xi_x, 0) \parallel_2 + \varepsilon \parallel R_2(\xi_x, 0) \parallel_2 \\ &+ \parallel \nu^{-\frac{1}{2}} \widehat{H_1(R_2)}(\xi_x, 0) \parallel_2 + \parallel (I-P_J)\hat{R}_2(\xi_x, 0) \parallel_2).\end{aligned}$$

So with C depending on ξ_x,

$$\begin{aligned}\parallel \hat{R}_{2v_y}(\xi_x, 0) \parallel_2 &\leq C(\frac{1}{\varepsilon}(\sum_{\xi_z} \parallel \nu^{-\frac{1}{2}} L_J(\widehat{I-P_J}) R_2(\xi_x, \xi_z) \parallel_2^2)^{\frac{1}{2}} \\ &+(\sum_{\xi_z} \parallel \nu^{-\frac{1}{2}} \widehat{H_1(R_2)}(\xi_x, \xi_z) \parallel_2^2)^{\frac{1}{2}} + \varepsilon(\sum_{\xi_z} \parallel R_2(\xi_x, \xi_z) \parallel_2^2)^{\frac{1}{2}} \\ &+\frac{1}{\varepsilon} \parallel \nu^{-\frac{1}{2}} L_J(\widehat{I-P_J}) R_2(\xi_x, 0) \parallel_2 + \varepsilon \parallel R_2(\xi_x, 0) \parallel_2 \\ &+ \parallel \nu^{-\frac{1}{2}} \widehat{H_1(R_2)}(\xi_x, 0) \parallel_2 + \parallel (I-P_J)\hat{R}_2(\xi_x, 0) \parallel_2),\end{aligned}$$

and similarly for $\hat{R}_{2v_y}(\xi_x, \xi_z)$ when $\xi_z \neq 0$.
For the ψ_4-moment we shall use

$$\hat{R}_{2v_z^2\bar{A}}(\xi) = \frac{1}{\sqrt{6}} \hat{R}_{24}(\xi) \int v_z^2 v^2 \bar{A} M dv + \hat{R}_{2\perp v_z^2 \bar{A}}(\xi). \tag{2.44}$$

Here in the 2-d case,

$$\begin{aligned}\xi_z \hat{R}_{2v_z^2\bar{A}}(\xi) &= -\xi_x \hat{R}_{2v_z v_x \bar{A}}(\xi) + (-1)^{\xi_z} i r_{v_z^2 \bar{A}} + i\varepsilon G \int \frac{\partial}{\partial v_z}(v_z \bar{A}) M \hat{R}_2(\xi) dv \\ &-\frac{i}{\varepsilon}(v_z^2 \bar{A}, \widehat{L_J R_2}(\xi)) - i(v_z \bar{A}, \widehat{H_1(R_2)}(\xi)).\end{aligned}$$

The additional term in comparison with the 1-d case, belongs to $R_{2\perp}$, and can be estimated by (2.29). An estimate of the boundary term $r_{v_z^2 \bar{A}}$ can be obtained by multiplying (2.42) with $v_z \bar{A}$ for $\xi_z = 0$ and integrating. For $\xi_z \neq 0$ this gives

$$|\hat{R}_{2v_z^2\bar{A}}(\xi_x, \xi_z)|^2 \leq C\big(\frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J) R_2 \parallel_{2,2}^2 + \varepsilon^2 \parallel R_2 \parallel_{2,2}^2 \big)$$

with C depending on ξ. Using (2.44) the same estimate holds for $\hat{R}_{24}(\xi_x, \xi_z)$. For $\xi_x \neq 0$ the Fourier component $\hat{R}_{2v_z^2\bar{A}}(\xi_x, 0)$ can be expressed by (2.35) and (2.42), including the pairwise cancellation of the $r(\xi_x)$ terms. Treating the Δ-term as in the 1-d case, gives

$$|\hat{R}_{2v_z^2\bar{A}}(\xi_x, 0)|^2 \leq C\big(\frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J) R_2 \parallel_{2,2}^2 + \varepsilon^2 \parallel R_2 \parallel_{2,2}^2 + \parallel \hat{f}^-(\xi_x) \parallel_{2\sim}^2 \big)$$

with C depending on ξ_x. Again by (2.44) the same estimate holds for $\hat{R}_{24}(\xi_x, 0)$.

The ψ_0 moments when $\xi_z \neq 0$ may now be obtained by multiplying (2.42) with v_z and integrating. Arguing as above and using the earlier estimate for the ψ_4-moment we get

$$|\hat{R}_{20}(\xi_x, \xi_z)|^2 \leq C\big(\frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I - P_J)R_2 \parallel_{2,2}^2 + \varepsilon^2 \parallel R_2 \parallel_{2,2}^2 + \parallel \hat{f}^-(\xi_x) \parallel_{2\sim}^2 \big)$$

with C depending on ξ. And so there only remains the ψ_0 moment for $\hat{R}_2(\xi_x, 0)$ when $\xi_x \neq 0$. Multiply (2.42) for $\xi = (\xi_x, 0)$ with v_x and integrate. For the boundary term $r_{v_x v_z}(\xi_x)$, multiply (2.42) for $\xi = (\xi_x, 1)$ with v_x and integrate. All terms in the upcoming expression for $r_{v_x v_z}(\xi_x)$ are then under control. This gives

$$|\hat{R}_{20}(\xi_x, 0)|^2 \leq c(\frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I - P_J)R_2 \parallel_{2,2}^2 + \varepsilon^2 \parallel R_2 \parallel_{2,2}^2 + \parallel \hat{f}^-(\xi_x) \parallel_{2\sim}^2).$$

Combining all the above estimates gives the statement of the lemma,

$$\parallel P_J R_2 \parallel_{2,2}^2 \leq \frac{c}{\varepsilon^2} \parallel f^- \parallel_{2,2,\sim}^2 . \quad \square$$

The step from L^2 to L^∞ for R_2 follows as in the R_1-case. These estimates together give

Lemma 2.6 *A solution to the R_2-problem satisfies*

$$\begin{aligned}
&\parallel \nu^{\frac{1}{2}}(I - P_J)R_2 \parallel_{2,2}^2 \leq c\Big(\varepsilon \parallel \nu^{-\frac{1}{2}}(I - P_J)g \parallel_{2,2}^2 + \frac{1}{\varepsilon} \parallel P_J g \parallel_{2,2}^2 \\
&\qquad\qquad + \frac{1}{\varepsilon^2} \parallel \bar{\psi} \parallel_{2,2,\sim}^2 \Big), \\
&\parallel P_J R_2 \parallel_{2,2}^2 \leq c\Big(\frac{1}{\varepsilon} \parallel \nu^{-\frac{1}{2}}(I - P_J)g \parallel_{2,2}^2 + \frac{1}{\varepsilon^3} \parallel P_J g \parallel_{2,2}^2 + \frac{1}{\varepsilon^4} \parallel \bar{\psi} \parallel_{2,2,\sim}^2 \Big), \\
&\parallel \nu^{\frac{1}{2}} R_2 \parallel_{\infty,2}^2 \leq c(\frac{1}{\varepsilon^3} \parallel \nu^{-\frac{1}{2}}(I - P_J)g \parallel_{2,2}^2 + \frac{1}{\varepsilon^5} \parallel P_J g \parallel_{2,2}^2 + \varepsilon^2 \parallel \nu^{-\frac{1}{2}} g \parallel_{\infty,2}^2 \\
&\qquad + \frac{1}{\varepsilon^6} \parallel \bar{\psi} \parallel_{2,\sim}^2).
\end{aligned}$$

The previous estimates can be used to prove

Theorem 2.2 *There exists a solution R in $L^2_M([-\pi, \pi]^2 \times \mathbb{R}^3)$ to the rest term problem*

$$v^\mu \cdot \nabla R - \varepsilon G M^{-1} \frac{\partial (MR)}{\partial v_z} = \frac{1}{\varepsilon} LR + J(R, R) + H(R) + \varepsilon \alpha, \qquad (2.45)$$

$$\begin{aligned}
R(x, \mp\pi, v) = &\int_{w_z \lessgtr 0} (R(x, \mp\pi, w) + \frac{1}{\varepsilon} \bar{\psi}(x, \mp\pi, w))|w_z| M_- dw \\
&- \frac{1}{\varepsilon} \bar{\psi}(x, \mp\pi, v), \quad v_z \gtrless 0,
\end{aligned}$$

Proof of Theorem 2.2 The rest term R will be obtained as the limit of the approximating sequence $\{R^n\}$, where $R^0 = 0$ and

$$v^\mu \cdot \nabla R^{n+1} - \varepsilon \frac{G}{M} \frac{\partial (MR^{n+1})}{\partial v_z} = \frac{1}{\varepsilon} L_J R^{n+1} + H_1(R^{n+1}) + J(R^n, R^n) + \varepsilon\alpha,$$

$$R^{n+1}(x, \mp\pi, v) = \frac{M_\mp}{M} \int_{w_z \lesseqgtr 0} (R^{n+1}(x, \mp\pi, w) + \frac{\bar\psi}{\varepsilon}(x, \mp\pi, w)) w_z M dw$$
$$- \frac{\bar\psi}{\varepsilon}(x, \mp\pi, v), v_z \gtrless 0.$$

Here $(I - P_J)g = \varepsilon(I - P_J)\alpha$ is of order four, and $P_J g = \varepsilon P_J \alpha$ of order five. In particular, the function R^1 is solution to

$$v^\mu \cdot \nabla R^1 - \varepsilon G M^{-1} \frac{\partial (MR^1)}{\partial v_z} = \frac{1}{\varepsilon} L_J R^1 + H_1(R^1) + \varepsilon\alpha,$$

$$R^1(x, \mp\pi, v) = \frac{M_\mp}{M} \int_{w_z \lesseqgtr 0} (R^1(x, \mp\pi, w) + \frac{\bar\psi}{\varepsilon}(x, \mp\pi, w)) |w_z| M_- dw$$
$$- \frac{\bar\psi}{\varepsilon}(x, \mp\pi, v), v_z \gtrless 0.$$

Splitting R^1 into two parts R_1 and R_2, solutions of (2.24) and (2.25) with $g = \varepsilon\alpha$ in (2.24), then using the corresponding a priori estimates, Lemma 2.3 and Lemma 2.6, together with the exponential decrease of $\bar\psi$, we obtain, for some constant c_1,

$$\| \nu^{\frac{1}{2}} R^1 \|_{\infty,2} \leq c_1 \varepsilon^{\frac{5}{2}}, \quad \| \nu^{\frac{1}{2}} R^1 \|_{2,2} \leq c_1 \varepsilon^{\frac{7}{2}}.$$

By induction for ε sufficiently small,

$$\| \nu^{\frac{1}{2}} R^j \|_{\infty,2} \leq 2c_1 \varepsilon^{\frac{5}{2}}, \quad j \leq n+1,$$
$$\| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2} \leq c_2 \varepsilon^2 \| \nu^{\frac{1}{2}} (R^n - R^{n-1}) \|_{2,2}, \quad n \geq 1,$$

for some constant c_2. Namely,

$$\frac{1}{\varepsilon} v^\mu \cdot \nabla (R^{n+2} - R^{n+1}) - G M^{-1} \frac{\partial (M(R^{n+2} - R^{n+1}))}{\partial v_z}$$
$$= \frac{1}{\varepsilon^2} L_J (R^{n+2} - R^{n+1}) + \frac{1}{\varepsilon} H_1(R^{n+2} - R^{n+1}) + \frac{1}{\varepsilon} G^{n+1},$$
$$(R^{n+2} - R^{n+1})(x, \mp\pi, v) = \frac{M_\mp}{M} \int_{w_z \lesseqgtr 0} (R^{n+2} - R^{n+1})(x, \mp\pi, w) |w_z| M_- dw,$$
$$v_z \gtrless 0.$$

Here, $G^{n+1} = (I - P)G^{n+1} = J(R^{n+1} + R^n, R^{n+1} - R^n)$. It follows that

$$\| \nu^{\frac{1}{2}} (R^{n+2} - R^{n+1}) \|_{2,2} \leq c\varepsilon^{-\frac{1}{2}} \| \nu^{-\frac{1}{2}} G^{n+1} \|_{2,2}$$
$$\leq c\varepsilon^{-\frac{1}{2}} \Big(\| \nu^{\frac{1}{2}} R^{n+1} \|_{\infty,2} + \| \nu^{\frac{1}{2}} R^n \|_{\infty,2} \Big) \| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2}$$
$$\leq c_2 \varepsilon^2 \| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2} .$$

Consequently,

$$\begin{aligned}\| \nu^{\frac{1}{2}} R^{n+2} \|_{2,2} &\leq \| \nu^{\frac{1}{2}}(R^{n+2}-R^{n+1}) \|_{2,2} + ... + \| \nu^{\frac{1}{2}}(R^2-R^1) \|_{2,2}\\ &+ \| \nu^{\frac{1}{2}} R^1 \|_{2,2} \leq 2c_1\varepsilon^{\frac{7}{2}},\end{aligned}$$

for ε small enough. Similarly $\| R^{n+2} \|_{\infty,2} \leq 2c_1\varepsilon^{\frac{5}{2}}$. In particular $\{R^n\}$ is a Cauchy sequence in $L^2_M([-\pi,\pi]^2 \times \mathbb{R}^3)$. The existence of a solution R to (2.45) follows. □

From here Theorem 2.1 follows, and as a consequence the first part of Theorem 1.1.

3 Stability: the expansion.

In the previous section we have constructed a stationary solution F_s of the Boltzmann equation close to the clockwise roll hydrodynamic solution h_s. In the next two sections we study the behavior in time of a small perturbation of F_s by writing the perturbation as a truncated ε-expansion and in particular in this section we show the decay to zero in time of the first terms of the expansion. This result relies crucially on the hydrodynamical stability under small perturbations of the hydrodynamic roll solution h_s. Hence, before starting the construction of the Boltzmann solution, let us recall some known hydrodynamic results. The Oberbeck-Boussinesq (O-B) equations with periodic and rigid boundary conditions (see [15]), describing the hydrodynamic behavior of the fluid in the present setup in dimensionless form, are:

$$\begin{aligned}&\partial_t u + u \cdot \nabla u = \hat{\eta} \triangle u - \nabla p - e_z G\theta,\\ &\frac{5}{2}(\partial_t \theta + u \cdot \nabla \theta + \lambda u_z) = \frac{5}{2}\hat{k}\Delta\theta,\\ &\text{div } u = 0.\end{aligned} \tag{3.1}$$

where e_z is the unit vector in he positive z-direction, $u \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ are the velocity field and the deviation from the linear temperature profile respectively, $\hat{\eta}$ and $\hat{k}$ are the dimensionless kinematic viscosity and conductivity respectively. The initial conditions are

$$u(x,z,0) = u_0(x,z), \quad \text{div } u_0 = 0, \quad \theta(x,z,0) = \theta_0(x,z)$$

for any $x, z \in \Omega_\mu = (-\mu\pi, \mu\pi) \times (-\pi, \pi)$. The boundary conditions for this problem are

$$\begin{aligned}&u(x,-\pi,t) = u(x,\pi,t) = \theta(x,-\pi,t) = \theta(x,\pi,t) = 0,\\ &x \in [-\pi,\pi], \quad t > 0.\end{aligned} \tag{3.2}$$

Here the notations are as in the Introduction. For a proof of the existence of a global in time solution for small initial data see for example [10]. The laminar solution is the trivial stationary solution $u = \theta = 0$. It is the unique solution for $Ra \leq Ra_c$ and is asymptotically stable for $Ra < Ra_c$ [12], [10]. After Ra_c a pair of new stationary solutions appear. In [12] and [13] it is proved that there exists δ_0 such that for any $0 < \delta \leq \delta_0$, there are two stationary roll solutions $(u_s^{\mp}, \theta_s^{\mp})$ corresponding to the Rayleigh number $Ra = Ra_c(1+\delta)$, of the form

$$u_s^{\mp}(x,z) = \mp\delta\ C_0\ \phi + O(\delta^2) \tag{3.3}$$

$$\theta_s^{\mp}(x,z) = \mp\delta\ C_0\ \tau + O(\delta^2).$$

Here C_0 is a positive constant, the couple (ϕ,τ) is the eigenfunction corresponding to the smallest eigenvalue d_0 of the linearized problem, namely the solution of

$$\begin{aligned}&\hat\eta\triangle\phi - \nabla p = e_z G\tau, \quad \text{div}\ \phi = 0, \quad \hat k\triangle\tau = d_0\phi_z,\\ &(\phi,\tau)(x,-\pi) = (\phi,\tau)(x,\pi) = 0, \quad (\phi,\tau)(x,z,t) = (\phi,\tau)(x+\mu\pi,z,t).\end{aligned}$$

In [14] both solutions are proved to be stable for small perturbations (see also [18]). All the previous results are stated in the Sobolev spaces H_2, but, by general theorems on PDE of parabolic type [16] (or by the method in [11]), the regularity can be improved to higher Sobolev spaces H_k. Hence we can state the stability theorem in a form suited to our purposes. Let $(u_s,\theta_s) \in (H_k)^3$, k large enough, be the clockwise solution of (3.3).

Theorem 3.1 *Let (u,θ) be the periodic solution of the following equation*

$$\begin{aligned}&\partial_t u + u_s\cdot\nabla u + u\cdot\nabla u_s + u\cdot\nabla u = \hat\eta\triangle u - \nabla p - e_z G\theta\\ &\frac{5}{2}(\partial_t\theta + u_s\cdot\nabla\theta + u\cdot\nabla\theta_s + \lambda u_z) = \frac{5}{2}\hat k\Delta\theta\\ &\text{div}\ u = 0,\\ &u(x,z,0) = u_0(x,z), \quad \theta(x,z,0) = \theta_0(x,z) \quad (x,z)\in[-\mu\pi,\mu\pi]\times[-\pi,\pi]\\ &u(x,-\pi,t) = u(x,\pi,t) = \theta(x,-\pi,t) = \theta(x,\pi,t) = 0, \quad x\in[-\pi,\pi], \quad t>0.\end{aligned}$$

If $(u_0,\theta_0) \in (H_k)^3$, k sufficiently large, and $\| u_0 \|_{H_k} + \| \theta_0 \|_{H_k} < n_0$, for n_0 small enough, then, $(u,\theta)(x,z,t)$ is in $(H_k)^3$ and $\lim_{t\to\infty}(u,\theta)(t,x,z) = 0$ exponentially in time in $(H_{k'})^3$, for any $k' < k$.

Notice that the convective solution $h_{conv} = (u_s, T_s)$ in the Introduction, (1.3) , is related to u_s, θ_s in (3.3) through the shift $T_s = \theta_s + \lambda(z+\pi)$.

In the previous section, we have constructed a positive stationary solution of the Boltzmann equation (1.1) as $F_s = M(1+\Phi_s^\varepsilon)$. Here, we want to study the evolution of positive perturbations of F_s. The perturbation Φ^ε, defined through $F = M(1+\Phi_s^\varepsilon + \Phi^\varepsilon)$, with F a solution to (1.1), has to solve the initial boundary value problem

$$\begin{aligned}
&\frac{\partial \Phi^\varepsilon}{\partial t} + \frac{1}{\varepsilon} v^\mu \cdot \nabla \Phi^\varepsilon - \frac{G}{M}\frac{\partial (M\Phi^\varepsilon)}{\partial v_z} = \frac{1}{\varepsilon^2}\Big(L\Phi^\varepsilon + J(\Phi^\varepsilon,\Phi^\varepsilon) + J(\Phi^\varepsilon_s,\Phi^\varepsilon)\Big),\\
&\Phi^\varepsilon(0,x,z,v) = \zeta_0(x,z,v), \quad (x,z)\in(-\pi,\pi)^2,\ v\in\mathbb{R}^3, \qquad (3.4)\\
&\Phi^\varepsilon(t,x,\pm\pi,v) = \frac{M_\pm}{M}\int_{w_z \gtrless 0} |w_z| M\Phi^\varepsilon(t,x,\pm\pi,w)dw,\\
&\qquad\qquad v_z \lessgtr 0,\ t>0,\ x\in[-\pi,\pi].
\end{aligned}$$

We consider the following initial perturbations

$$F(0,x,z,v) - F_s := \Phi^\varepsilon(0,x,z,v) = \zeta_0(x,z,v),$$

$$\zeta_0(x,z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(0,x,z,v) + \varepsilon^5 p_5, \qquad (3.5)$$

where

$$F(0,\cdot,\cdot,\cdot)\geq 0, \quad \| p_5 \|_{\infty,2} := \sup_{\varepsilon>0}\left(\int \sup_{(x,z)\in[-\pi,\pi]^2} p_5^2(x,z,v)Mdv\right)^{\frac{1}{2}} < c, \qquad (3.6)$$

for some constant c. The nonhydrodynamic part of the functions $\Phi^{(n)}(0,x,z,v)$ is determined by the expansion as explained below together with some terms of the hydrodynamic part. We will denote by $I_i^{(n)}(t,x,z)$ the coefficients of the functions ψ_i in the hydrodynamic part of $\Phi^{(n)}(t,x,z,v)$. The functions $I_i^{(1)}(t,x,z)$ will be determined by the solution (u,θ) in Theorem 3.1. Finally, we require

$$\int_{[-\pi,\pi]^2\times\mathbb{R}^3} \zeta_0(x,z,v)M\psi_0(v)dxdzdv = 0.$$

Since the Boltzmann equation conserves the total mass, it follows that Φ^ε will satisfy

$$\int_{[-\pi,\pi]^2\times\mathbb{R}^3} M\Phi^\varepsilon(x,z,v)\psi_0(v)dvdxdz = 0, \quad t>0.$$

We write an ε-expansion for Φ^ε in the form

$$\Phi^\varepsilon(t,x,z,v) = \sum_{n=1}^{5} \Phi^{(n)}(t,x,z,v)\varepsilon^n + \varepsilon R(t,x,z,v).$$

For the proof of stability we need to show that $\Phi^{(n)}(t,x,z,v)$ converge to zero, when time tends to infinity in a suitable norm. To this end we will construct explicitly the first terms of the expansions. The behavior of the higher order terms will then be evident from this analysis. This construction is by now standard and contained in many papers. We give here a sketch of the argument for sake of completeness, and follow closely the analysis in [1].

In the following we use the notation $\langle h\rangle = \int_{\mathbb{R}^3} dvh(v)$. By plugging the expansion into (3.4), as a first condition, $\Phi^{(1)}$ has to be a combination of the collision invariants $\psi_i, i = 0, \ldots, 4$

$$\Phi^{(1)} = \left(\rho^1 + u^1 \cdot v + \theta^1 \frac{|v|^2 - 3}{2}\right) ,$$

so that $\rho^1 \equiv I_0^{(1)}$, $u_x^1 \equiv I_1^{(1)}$, $0 \equiv I_2^{(1)}$, $u_z^1 \equiv I_3^{(1)}$, $\theta^1 \equiv \frac{2}{\sqrt{6}} I_4^{(1)}$. We require that u^1, θ^1 satisfy the initial and boundary conditions in Theorem 3.1 and, as a consequence, do not need boundary layer correction to the first order in ε. Indeed, in $z = -\pi$ the solution is already of the right type. On the other hand, $M + \varepsilon\Phi^{(1)}$, when evaluated for $z = \pi$, cannot satisfy the boundary conditions, but differs from it by terms of order ε^2, which will appear in the corrections of higher order. Hence, for $n > 1$ the higher order corrections are decomposed into a bulk term $B^{(n)}$ and two boundary layer terms $b_\pm^{(n)}$.

To determine the functions ρ^1, u^1 and θ^1 which give $\Phi^{(1)} (= B^{(1)})$, we consider the equation obtained by equating the terms of next order. Note from the previous sections that the stationary solution can also be expanded in ε, and denote by $\Phi_s^{(n)}$ the terms of this expansion. The equation which we get at next order, by ignoring boundary layer corrections, is

$$v_x \frac{\partial}{\partial x}\Phi^{(1)} + v_z \frac{\partial}{\partial z}\Phi^{(1)} = LB^{(2)} + J(\Phi^{(1)}, \Phi^{(1)}) + J(\Phi^{(1)}, \Phi_s^{(1)}) \tag{3.7}$$

It can be seen as an equation in $B^{(2)}$, whose solvability conditions give the usual incompressibility condition and the Boussinesq condition

$$\text{div } u = 0, \qquad \nabla(\theta^1 + \rho^1) = 0. \tag{3.8}$$

The Boussinesq condition fixes $\rho^1 = -\theta^1$, up to a constant. To determine θ^1 and u^1 we look at the solvability condition at next order in ε. Indeed, once (3.8) is satisfied, we can deduce from (3.7) the following expression for $B^{(2)}$, where L^{-1} denotes the inverse of the restriction of L to the orthogonal of its null space,

$$B^{(2)} = L^{-1}\left[v \cdot \nabla\Phi^{(1)} - J(\Phi^{(1)}, \Phi^{(1)}) - J(\Phi^{(1)}, \Phi_s^{(1)})\right] + \sum_{i=0}^{4} \psi_i \, I_i^{(2)}(t, z) \; . \tag{3.9}$$

The coefficients $I_i^{(2)}$ are undetermined at this point and will be partly fixed by the solvability condition for the equation at next order in ε and the rest of them in some later step,

$$\begin{aligned} \frac{\partial}{\partial t}B^{(1)} \; &+ v \cdot \nabla B^{(2)} - \frac{1}{M} G \frac{\partial}{\partial v_z}(MB^{(1)}) = LB^{(3)} + J(\Phi^{(2)}, \Phi^{(1)}) \\ &+ J(\Phi^{(2)}, \Phi_s^{(1)}) + J(\Phi^{(1)}, \Phi_s^{(2)}). \end{aligned} \tag{3.10}$$

The solvability conditions for this equation,

$$(\psi_i, [\frac{\partial}{\partial t}B^{(1)} + v \cdot \nabla B^{(2)} - \frac{1}{M} G \frac{\partial}{\partial v_z}(MB^{(1)})]) = 0, \quad i = 0, \cdots, 4, \tag{3.11}$$

produce the equations for u^1 and θ^1. Let us fix $i = 1,2,3$ in (3.11). Then the first term gives the time derivative of u^1. The third one reduces to 0 for $i = 1,2$, and to $-G\rho^1$ for $i = 3$ after integrating by parts. The term $\langle v \otimes vB^{(2)} \rangle$ gives rise to dissipative transport terms and a term which can be interpreted as the second order correction to the pressure P_2. The term $J(\Phi^{(1)}, \Phi_s^{(1)})$ in (3.9) produces the linear transport terms, depending on the stationary flow. The result is

$$\frac{\partial}{\partial t} u^1 + u^1 \cdot \nabla u^1 + u^1 \cdot \nabla u_s^1 + u_s^1 \cdot \nabla u^1 = \hat{\eta} \triangle u_z^1 - \nabla P_2 + e_z G \rho^1 \ .$$

Using the Boussinesq condition we replace the term $G\rho^1$ by $-G\theta^1 + const.$ The constant can be absorbed in the pressure term that we rename p.

Remark. There are constants (one coming from the Boussinesq condition, another from the pressure condition) at any order which will be determined in the end by the total mass condition. Since we are asking that the total mass of the perturbation is zero we can put to zero all the constants.

To get the equation for the temperature, one has to look at (3.11) for $i = 4$. It is actually more convenient to replace ψ_4 with the equivalent $\tilde{\psi}_4 = \frac{1}{2}(v^2 - 5)$. We have

$$\begin{aligned} (\tilde{\psi}_4, f_1) &= \frac{5}{2}\theta^1, \quad G(\tilde{\psi}_4, \frac{\partial}{\partial v_z} f_1) = -u_z^1 G, \\ (v\tilde{\psi}_4, B^{(2)}) &= -\frac{5}{2}\hat{k}\nabla\theta^1 + \frac{5}{2}u^1\theta^1 + u_s^1\theta^1 + \theta_s^1 u^1 \ . \end{aligned}$$

Putting all the terms together, we get

$$\frac{5}{2}\left[\frac{\partial}{\partial t}\theta^1 + u^1\nabla\theta^1 + u_s^1\nabla\theta^1 + u^1\nabla\theta_s^1\right] - G u_z^1 = \frac{5}{2}\hat{k}\triangle\theta^1 \ .$$

This equation has to be solved with boundary conditions $\theta^1(\pm 1, t) = 0$ for $t > 0$, and an initial condition θ_0^1, which is completely arbitrary.

Remark. In the previous equation there is a term $-G u_z^1$ which does not appear in the usual O-B equations. This term can be absorbed by changing the boundary conditions. Here θ_s^1, u_s^1 are the hydrodynamic terms of first order in ε in the expansion of Φ_s, and hence they coincide with u_s, T_s in (1.3). The boundary conditions for T_s are: $T_s(x, -\pi, t) = \ 0, T_s(x, \pi, t) = 2\lambda\pi$. The shifted temperature $\tilde{T}_s = T_s - Gz$ will satisfy the usual Boussinesq equation, in which the term Gu_z is missing and a different boundary condition, $T_s(x, \pi, t) = (2\lambda - G)\pi$. This aspect was discussed in [9]. It was pointed out that starting from the compressible Navier-Stokes equation or the Boltzmann equation in the scaling we are considering, one obtains a set of equations which differ from the usual O-B ones for this shift in the boundary condition for the temperature. By scaling the variables, this amounts to the usual O-B equations in dimensionless form, with a new Rayleigh number given by $Ra(1 - G)$. We conclude that nothing changes in our analysis.

To summarize what we got so far, (u^1, θ^1) has to satisfy the O-B equations in Theorem 3.1. By fixing the initial conditions so that the assumptions of the Theorem are satisfied, we get that (u^1, θ^1) vanishes exponentially in time, with its spatial derivatives. Since θ^1 differs from ρ^1 by a constant, which can be taken as zero, we may conclude that for $q \in [1, +\infty]$, $\| \Phi^{(1)} \|_{q,2}$ is finite and converges to zero exponentially in time.

The second order term in the expansion, $\Phi^{(2)}$, is not yet completely determined. Equation (3.11) with $i = 0$ gives $\frac{\partial}{\partial t}\rho^1 = \text{div } I^{(2)}$, fixing div $I^{(2)}$. Moreover, a combination of $I_0^{(2)}$ and $I_4^{(2)}$ contributes to the pressure p which is determined by the previous equations, so that these parameters are not independent.

The nonhydrodynamic part of $B^{(2)}$ is a linear function of the derivatives of ρ^1, θ^1 which are in general different from zero at the boundaries. Therefore the non hydrodynamical part of $B^{(2)}$ is completely fixed (even at time zero) and violates the boundary conditions. We need to introduce $b_\pm^{(2)}$ to restore the boundary conditions by compensating the non hydrodynamical part of $B^{(2)}$ which is not Maxwellian. We explain how to find the correction $b_-^{(2)}$. The correction $b_+^{(2)}$ is found in a similar way. Here $L^- = 2M_-^{-1}Q(M_-, M_- \,\cdot)$. We choose $b_-^{(2)}$ by solving, for any $t > 0$, the Milne problem for $z^- > 0$,

$$v_z \frac{\partial}{\partial z^-} h - \varepsilon^2 \frac{1}{M} G^- \frac{\partial}{\partial v_z}(Mh) = L^- h, \quad \langle v_z h \rangle := \int_{\mathbb{R}^3} dv v_z h = 0, \tag{3.12}$$

where $z^- = \varepsilon^{-1}(z + \pi)$ is defined as the rescaled z variable near the bottom plate, and G^- is a smooth force rapidly decaying to zero far from the bottom plate. Indeed, the gravity force has been decomposed in three parts, a force constant in the bulk and two boundary parts $G^\pm$ (see [9], [8] for details). We impose the boundary condition at $z^- = 0$ in such a way that the incoming flux of h at $z = -\pi$, $v_z > 0$, is given by $(I - P)B^{(2)}(-1, v; t)$. The results in [6] tell us that as $z^- \to +\infty$ the solution approaches a function $q_-^{(2)}(v, t)$ in `Kern` L^-. Note that in $q_-^{(2)}$ there is no term proportional to ψ_3 because of the vanishing mass flux condition in the direction of the z axis $\langle v_z h \rangle = 0$. Thus we set $b_-^{(2)}(x, z_-, v, t) = h(x, z_-, v, t) - q_-^{(2)}(x, v, t)$, which will go to zero at infinity exponentially in z^-. This produces a term $b_-^{(2)}(x, 2\pi\varepsilon^{-1}, v, t) = \psi_{2,\varepsilon}(x, \pi, v, t)$, exponentially small in ε^{-1} on the opposite boundary. Scaling again to the variable z, the resulting term in the expansion is thus $\Phi^{(2)} = B^{(2)} + b_+^{(2)} + b_-^{(2)}$, and is such that in $z = -\pi$, for example, it has zero non hydrodynamic part, while the hydrodynamic part is

$$\Phi^{(2)}(x, -\pi, v; t) = \sum_{i=0}^{4} I_i^{(2)}(x, -\pi; t)\psi_i(v) + b_+^{(2)}(x, 2\varepsilon^{-1}, v, t) - q_-^{(2)},$$

$$v_z > 0, \quad t > 0.$$

We are not yet done since $M\Phi^{(2)}(x, -\pi, v)$ is not Maxwellian for $v_z > 0$, (as it should, in order to satisfy the boundary conditions) because of the presence of terms proportional to ψ_i, $i = 1, 2, 4$ in $q_-^{(2)}$ and $b_+^{(2)}(t, x, 2\varepsilon^{-1})$. The latter is not important because it is a correction exponentially small in ε and will be compensated in the remainder. The former will be compensated by the coefficients $I_i^{(2)}$, $i \neq 0, 3$, that can be chosen arbitrarily on the boundaries. Moreover, we have to choose $I_3^{(2)} = 0$ on the boundaries, because $\langle v_z q_-^{(2)} \rangle = 0$. We are left with

$$\Phi^{(2)}(t, x, \pm\pi, v_z \lesseqgtr 0) = \alpha_2^{\pm} M_{\pm} + \psi_{2,\varepsilon}(x, \pm\pi, v, t), \quad \alpha_2^{\pm} = I_0^{(2)}(\pm 1) - \langle q_{\pm}^{(2)} \rangle,$$

where $\psi_{2,\varepsilon}$ are terms exponentially small in ε. Finally, we impose the impermeability condition $\langle v_z \Phi^{(2)} \rangle = 0$ by choosing

$$\alpha_2^{\pm} = \frac{M_{\pm}}{M} \int_{v_z \gtrless 0} v_z M[\Phi^{(2)}(t, x, \pm\pi, v) - \psi_{2,\varepsilon}(t, x, \pm\pi, v)] dv, \quad v_z \lesseqgtr 0, t > 0.$$

The coefficients $I_i^{(2)}, i = 1, 2, 4$ of the hydrodynamical part of $B^{(2)}$ are determined by the compatibility condition for the equation at next order in ε,

$$\Big(\psi_i, \big[\frac{\partial}{\partial t} B^{(2)} + v \cdot \nabla B^{(3)} + G \frac{\partial}{\partial v_z} B^{(2)}\big]\Big) = 0,$$

where

$$B^{(3)} = L^{-1}\Big[\frac{\partial}{\partial t}\Phi^{(1)} + v \cdot \nabla B^{(2)} + \frac{1}{M} G \frac{\partial}{\partial v_z}(M\Phi^{(1)}) - J(\Phi^{(1)}, B^{(2)})$$
$$-J(\Phi_s^{(1)}, B^{(2)}) - J(\Phi^{(1)}, B_s^{(2)})\Big] + \sum_{i=0}^{4} \psi_i \; I_i^{(3)},$$

together with the boundary conditions $I_i^{(2)} = (q_-^{(2)})_i, \; i = 1, 2, 4$. Then $I_0^{(2)}$ is found up to a constant that is chosen so that the total mass associated to $\Phi^{(2)}$ vanishes. Proceeding as in the determination of the Boussinesq equation, we find now a set of three linear time-dependent nonhomogeneous Stokes equations for $I_i^{(2)}$,

$$\begin{aligned}
&\partial_t \rho^2 + \text{div } u^2 + \text{div } (\rho^1 u^1) + \text{div } (u^3) = N_0 \\
&\partial_t u^2 = u^2 \cdot \nabla u^1 + u^1 \cdot \nabla u^2 = \hat{\eta} \triangle u^2 - \nabla P^3 + G\rho^2 + \nabla \text{div } u^2 + N \\
&\partial_t \theta^2 + \frac{2}{3}[\text{div } u^3 + (\rho^1 + \theta^1)\text{div } u^2] + \rho^1[\partial_t \theta^1 + \frac{2}{3}\text{div } u^2] \\
&\qquad = \hat{k}\frac{2}{3}[\triangle \theta^2 + (\nabla u_1)^2] + N_4,
\end{aligned}$$

where N_0, N_4 depend on the third order spatial derivatives of ρ^1, θ^1 and N depend on the third order spatial derivatives of u^1. We remember that P^2 is determined by p which has been found at the previous step by solving the O-B for u^1, θ^1. On the other hand, $P^2 = \rho^2 + \theta^2 + \rho^1\theta^1$ allows us to eliminate ρ^2 from the previous equations. Replacing div (u^3) as given from the first equation in the last one, and using the condition on P^2, we get a set of two coupled equations for θ^2, u^2 with a constraint on divu^2. P^3 plays the role of Lagrangian multiplier for this constraint. The nonhomogeneous term is controlled by the results at the previous step and hence is known to decay to zero exponentially in time in the right norms. Then, general theorems for the

Stokes equation assures the existence of a solution for the chosen boundary conditions, vanishing exponentially in time.

Once $B^{(2)}$ is completely determined, the last equation gives the non-hydrodynamical part of $B^{(3)}$. As before, we introduce the terms $b_{\pm}^{(3)}$ to compensate $(I-P)B^{(3)}$ on the boundaries $z=\pm\pi$. The term $b_{\pm}^{(3)}$ is found as a solution of a Milne problem with a source term, which depends on the previous boundary corrections $b_{\pm}^{(2)}$ and $\Phi^{(1)}$. The procedure can be continued to any order.

We notice that $(I-P)\Phi^{(n)}$ at time zero are not arbitrary, since they depend on $\Phi^{(n-1)}$ and its derivatives. We can instead assign at time zero $I_i^{(n)}, i=1,2,4$. Notice that the rest term R at time zero is of order ε^4. By using the results in [6] and the exponential decay in time of $\Phi^{(n)}$ we can state the following theorem

Theorem 3.2 *Assume that at time zero, for some suitably large k,*

$$\parallel M\partial^k I_i^{(n)}(0,x,z)\parallel_{L^2}<\infty,\quad i=1,2,4,\quad n=1,\dots 5\ ,$$

where ∂^k denotes any space derivative of order k. Then, it is possible to determine the functions $\Phi^{(n)}$, $n=2,\dots,5$ in the asymptotic expansion satisfying the boundary conditions

$$\begin{aligned}&\Phi^{(n)}(t,x,\mp\pi,v)=\\&\frac{M_\pm(v)}{M(v)}\int_{w_z\lesseqgtr 0}|w_z|M\big[\Phi^{(n)}(t,x,\mp\pi,w)-\psi_{n,\varepsilon}(t,x,\mp\pi,w)\big]dw\\&+\psi_{n,\varepsilon}(t,x,\mp\pi,v),\ t>0,\ v_z\gtreqless 0,t>0,\end{aligned}$$

the normalization condition $\displaystyle\int_{\mathbb{R}^3\times\Omega_\mu} M\Phi^{(n)}dvdzdx=0,\quad t\in\mathbb{R}^+$, *and*

$$\parallel\Phi^{(n)}\parallel_{2,2,2}<\infty,\quad\parallel\Phi^{(n)}\parallel_{\infty,\infty,2}<\infty\ .$$

Here,

$$\parallel f\parallel_{2,2,2}=\Big(\int_0^\infty\int_{\Omega_\mu}\int_{\mathbb{R}^3}|f(s,x,z,v)|^2M(v)dsdxdzdv\Big)^{\frac12},$$

$$\parallel f\parallel_{\infty,\infty,2}=\sup_{t>0}\Big(\int_{\mathbb{R}^3}\sup_{(x,z)\in\Omega_\mu}|f(t,x,z,v)|^2M(v)dv\Big)^{\frac12}\ .$$

4 Stability: the remainder

We remember that Φ^ε, the solution to (3.4), is written as $\Phi^\varepsilon = \sum_{i=1}^5 \varepsilon^i \Phi^{(i)} + \varepsilon R$. In the previous section we have constructed the terms $\Phi^{(i)}$ and shown that they decay to zero in suitable norms. In this section we construct the rest term R, solution of

$$\begin{aligned}
&\frac{\partial R}{\partial t} + \frac{1}{\varepsilon}\mu\, v_x \frac{\partial R}{\partial x} + \frac{1}{\varepsilon} v_z \frac{\partial R}{\partial z} - GM^{-1}\frac{\partial (MR)}{\partial v_z} = \frac{1}{\varepsilon^2} LR + \frac{1}{\varepsilon} J(R,R) \\
&\qquad\qquad + \frac{1}{\varepsilon} H(R) + A, \qquad (4.1)\\
&R(0,x,z,v) = R_0(x,z,v) = \varepsilon^4 p_5(x,z,v),\\
&R(t,x,\mp\pi,v) = \frac{M_\mp}{M}\int_{w_z \lesseqgtr 0} (R(t,x,\mp\pi,w) + \frac{\bar\psi}{\varepsilon}(t,x,\mp\pi,w))|w_z| M dw\\
&\qquad - \frac{\bar\psi}{\varepsilon}(t,x,\mp\pi,v), \qquad x \in [-\pi,\pi], \quad t>0,\ v_z>0,
\end{aligned}$$

Here $\bar\psi(t,x,\pm\pi,v) = \sum_n \varepsilon^n \psi_{n,\varepsilon}(t,x,\pm\pi,v)$ is the Knudsen part of the asymptotic expansion from $(t,x,\mp\pi,v)$, exponentially small when evaluated at $(t,x,\pm\pi,v)$. A contains all the terms fully coming from the asymptotic expansion,

$$H(R) = \frac{1}{\varepsilon} J(R, \bar\Phi + \Phi_s),$$

where $\bar\Phi = \sum_1^5 \Phi^{(j)}\varepsilon^j$, and $\int p_5 M dxdzdv = 0$. We shall require that the initial value of Φ^ε is close to zero, and in the sequel introduce smallness assumptions.

The following norms will be used,

$$\begin{aligned}
\| R \|_{2t,2,2} &= \Big(\int_0^t \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^3} R^2(s,x,z,v)M(v)dsdxdzdv\Big)^{\frac12},\\
\| R \|_{\infty,2,2} &= \sup_{t>0}\Big(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^3} R^2(t,x,z,v)M(v)dxdzdv\Big)^{\frac12},\\
\| R \|_{\infty,\infty,2} &= \sup_{t>0}\Big(\int_{\mathbb{R}^3} \sup_{-\pi<x,z<\pi} R^2(t,x,z,v)M(v)dv\Big)^{\frac12},
\end{aligned}$$

$$\begin{aligned}
\| f \|_{2t,2,\sim} &= \Big(\int_0^t \int_{-\pi}^{\pi}\int_{v_z>0} v_z M(v) \mid f(s,x,-\pi,v)\mid^2 dvdxds\Big)^{\frac12}\\
&\quad + \Big(\int_0^t \int_{-\pi}^{\pi}\int_{v_z<0} \mid v_z\mid M(v) \mid f(s,x,\pi,v)\mid^2 dvdxds\Big)^{\frac12},\\
\| f \|_{\infty,2,\sim} &= \Big(\sup_{t>0}\int_{-\pi}^{\pi}\int_{v_z>0} v_z M(v) \mid f(t,x,-\pi,v)\mid^2 dxdv\Big)^{\frac12}\\
&\quad + \Big(\sup_{t>0}\int_{-\pi}^{\pi}\int_{v_z<0} \mid v_z\mid M(v) \mid f(t,x,\pi,v)\mid^2 dxdv\Big)^{\frac12}.
\end{aligned}$$

We will prove the existence of Φ^ε and the stability result (1.7). We follow closely the approach in Section 2, starting from dual, space-periodic solutions to a linear problem (in the rescaled time variable $\bar\tau = \varepsilon^{-1}t$) discussed in the following lemma. We use the notations introduced in Section 2, $L_J = L(\cdot)+\varepsilon J(q, P\cdot)$, but here the function q has the expression $q = \varepsilon^{-1}(\bar\Phi + \Phi_s)$ which is also time dependent.

Lemma 4.1 *Let $\varphi(\bar\tau, x, z, v)$ be solution to*

$$\frac{\partial\varphi}{\partial\bar\tau} + \mu\ v_x\frac{\partial\varphi}{\partial x} + v_z\frac{\partial\varphi}{\partial z} - \varepsilon GM^{-1}\frac{\partial(M\varphi)}{\partial v_z} = \frac{1}{\varepsilon}L_J^*\varphi + g, \tag{4.2}$$

periodic in x of period 2π, with zero initial and ingoing boundary values at $z = -\pi, \pi$, and g x-periodic of period 2π. Set $\tilde\varphi = \varphi - <\varphi> = \varphi - (2\pi)^{-2}\int\varphi dxdz$.

Then, if $\varepsilon \le \varepsilon_0$, $\delta \le \delta_0$, for ε_0, δ_0 small enough, there exists η small such that,

$$\begin{aligned}
\parallel \varphi \parallel_{\infty,2,2} &\le c\Big(\varepsilon^{\frac12} \parallel \nu^{-\frac12}(I-P)g \parallel_{2,2,2} + \varepsilon^{-\frac12} \parallel Pg \parallel_{2,2,2} \\
&\quad + \eta\varepsilon^{\frac12} \parallel < P\varphi > \parallel_{2,2}\Big), \\
\parallel \nu^{\frac12}(I-P)\varphi \parallel_{2,2,2} &\le c\Big(\varepsilon \parallel \nu^{-\frac12}(I-P)g \parallel_{2,2,2} + \parallel Pg \parallel_{2,2,2} \\
&\quad + \eta\varepsilon \parallel < P\varphi > \parallel_{2,2}\Big), \\
\parallel \widetilde{P\varphi} \parallel_{2,2,2} &\le c\Big(\parallel \nu^{-\frac12}(I-P)g) \parallel_{2,2,2} + \varepsilon^{-1} \parallel Pg \parallel_{2,2,2} \\
&\quad + \eta \parallel < P\varphi > \parallel_{2,2}\Big).
\end{aligned}$$

Proof of Lemma 4.1. A variant of the method in [Ma Scn 7.3] can be adapted to the present setting with a force term, to obtain the existence of a solution to (4.2).

Denote by $\hat\varphi(\bar\tau, \xi, v)$, $\xi = (\xi_x, \xi_z) \in \mathbb{Z}^2$ the Fourier transform of φ with respect to space, and define $\hat g$ analogously. Then for $\xi \neq (0, 0)$,

$$\frac{\partial\hat\varphi}{\partial\bar\tau} = \frac{1}{\varepsilon}\widehat{L_J^*\varphi} - i\xi\cdot v^\mu\ \hat\varphi + \varepsilon GM^{-1}\frac{\partial(M\hat\varphi)}{\partial v_z} + \hat g - |v_z|r(-1)^{\xi_z}.$$

Here $v^\mu = (\mu v_x, v_z)$, $r = \mathcal{F}_x\varphi(\bar\tau, \xi_x, \pi, v)$ for $v_z > 0$, $r = \mathcal{F}_x\varphi(\bar\tau, \xi_x, -\pi, v)$ for $v_z < 0$, with $\mathcal{F}_x$ denoting Fourier transform with respect to the x-variable. Let β be a truncation function belonging to $C^1(\mathbb{R})$ with support in $(0, \infty]$, and such that $\beta(\bar\tau) = 1$ for $\bar\tau > \tau_0$ for some $\tau_0 > 0$. Let $\bar{\hat\varphi} = \hat\varphi\beta$. Then, for $(0, 0) \neq \xi \in Z^2$,

$$\frac{\partial\bar{\hat\varphi}}{\partial\bar\tau} = \frac{1}{\varepsilon}\widehat{L_J^*\beta\varphi} + i\xi\cdot v^\mu\bar{\hat\varphi} + \varepsilon GM^{-1}\frac{\partial(M\bar{\hat\varphi})}{\partial v_z} + \hat\varphi\frac{\partial\beta}{\partial\bar\tau} + \hat g\beta - |v_z|r\beta(-1)^{\xi_z}.$$

Let $\mathcal{F}$ be the Fourier transform in $\bar\tau$ with Fourier variable σ. We put

$$\Phi = \mathcal{F}\bar{\hat\varphi},\ \tilde Z = \mathcal{F}\left(\varepsilon^{-1}\widehat{L_J^*\beta\varphi} + \varepsilon GM^{-1}\frac{\partial(M\bar{\hat\varphi})}{\partial v_z} + \hat\varphi\frac{\partial\beta}{\partial\bar\tau} + \hat g\beta - |v_z|r\beta(-1)^{\xi_z}\right),$$

$$Z = \mathcal{F}\left(\varepsilon^{-1}\widehat{L_J^*\beta\varphi} + \hat{g}\beta - |v_z|r\beta(-1)^{\xi_z}\right),$$
$$Z' = \mathcal{F}\left(\varepsilon^{-1}\widehat{L_J^*\beta\varphi} + \hat{g}\beta\right),\ \hat{U} = (i\sigma + i\xi\cdot v^{\mu})^{-1}.$$

Let χ be the indicatrix function of the set $\{v;\ \mid\sigma+\xi\cdot v^{\mu}\mid<\alpha|\xi|\}$, for some positive α to be chosen later. Similarly to Section 2, the elements $\bar{\psi}_0, ..., \bar{\psi}_4$ are an orthonormal basis for the kernel of L_J^*. Let $\zeta_s(v) = (1+\mid v\mid)^s$. For $\xi \neq (0,0)$

$$\begin{aligned}\| P(\chi\Phi) \| &\leq c\sum_{j=0}^{4}\left|\int \chi\Phi(\sigma,\xi,v)\bar{\psi}_j M dv\right| \ \| \bar{\psi}_j \| \\ &\leq c \| \zeta_{-s}\chi\Phi \| \sum_{j=0}^{4} \| \chi\zeta_s\bar{\psi}_j \| \leq c\sqrt{\alpha} \| \zeta_{-s}\chi\Phi \| .\end{aligned}$$

Use this estimate on the support of χ for $\alpha = \| \zeta_{-s}\Phi \|^{-1} \| \zeta_{-s}Z' \|$.

As in Section 2, the previous estimate also holds with respect to supp χ_1 where the indicatrix function χ_1 is taken for $\alpha = \sqrt{\delta_1}$. We fix δ_1 so that $c\sqrt{\delta}_1 << 1$. Then the above estimate gives that the hydrodynamic P-part of the right-hand side, $\| P(\chi_1\Phi) \|$, can be absorbed by $\| P(\chi_1\Phi) \|$ in the left-hand side. The estimates hold in the same way when χ_1 is suitably smoothed around $\sqrt{\delta_1}|\xi|$. For the remaining $(1-\chi)(1-\chi_1)\Phi = \chi^c\chi_1^c\Phi$ we shall use that $\Phi = -\hat{U}\tilde{Z}$. Then

$$\begin{aligned}&\| P\chi^c\chi_1^c\Phi \|^2 \leq c\Big(\| \zeta_s\chi^c\chi_1^c\hat{U} \|^2 + \| \zeta_{s+2}\chi^c\chi_1^c\hat{U} \|^2 \Big) \| \zeta_{-s}Z' \|^2 + \tilde{\Theta} \\ &+\frac{C \| \mathcal{F}(\sqrt{|v_z|}\beta r) \|^2}{\delta_1|\xi|^2} \leq \frac{C}{\mid\xi\mid^2\mid\alpha\mid} \| \zeta_{-s}Z' \|^2 + \frac{\| \mathcal{F}(\sqrt{|v_z|}\beta r) \|^2}{\delta_1|\xi|^2} + \tilde{\Theta}\end{aligned}$$

where

$$\begin{aligned}\tilde{\Theta} := -2\sum_{j=0}^{4}\int \bar{\psi}_j\chi^c\chi_1^c\hat{U}\mathcal{F}\Big(\varepsilon G M^{-1}\frac{\partial(M\bar{\hat{\varphi}})}{\partial v_z} + \hat{\varphi}\frac{\partial\beta}{\partial\bar{\tau}}\Big)M dv \\ \times\Big(\int \bar{\psi}_j\chi^c\chi_1^c(\mathcal{F}\hat{\varphi}\beta - \hat{U}Z)M dv\Big)^*.\end{aligned}$$

We again replace α in the denominator by $\| \zeta_{-s}\Phi \|^{-1} \| \zeta_{-s}Z' \|$. That gives

$$\begin{aligned}\| P\Phi \|^2 &\leq c(\| \zeta_{-s}\Phi \| \| \zeta_{-s}Z' \| + \frac{\| \mathcal{F}(\sqrt{|v_z|}\beta r) \|^2}{\delta_1|\xi|^2} \\ &\quad + \sqrt{\delta_1} \| \zeta_{-s}(I-P)\Phi \|^2) + \tilde{\Theta}.\end{aligned}$$

Hence,

$$\begin{aligned}\| P\Phi \|_H^2 &\leq C\Big((\| P\Phi \| + \| \zeta_{-s}(I-P)\Phi \|) \| \zeta_{-s}Z' \| + \frac{\| \mathcal{F}\sqrt{|v_z|}\beta r \|^2}{\delta_1|\xi|^2} \\ &\quad + \sqrt{\delta_1} \| \zeta_{-s}(I-P)\Phi \|^2 \Big) + \tilde{\Theta}.\end{aligned}$$

Consequently,

$$\begin{aligned}\parallel P\Phi \parallel^2 \ &\le c\Big(\parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel \mathcal{F}(\sqrt{|v_z|}\beta r) \parallel^2}{\delta_1|\xi|^2} + \parallel \zeta_{-s}(I-P)\Phi \parallel\parallel \zeta_{-s}Z' \parallel \\ &\quad + \parallel \zeta_{-s}(I-P)\Phi \parallel^2 \Big) + \tilde{\Theta}.\end{aligned}$$

We next discuss the term $\tilde{\Theta}$. The first term in the first integral can be bounded by ε times an integral of a product of M, $1+|\xi_z|$, a polynomial in v, $|\ \mathcal{F}\bar{\tilde{\varphi}}\ |$ and $\hat{U}$ or $\hat{U}^2$. So this factor is bounded by $\varepsilon C \parallel \Phi \parallel$. And so,

$$\begin{aligned}&\parallel P\Phi \parallel^2 \le C\Big(\parallel \zeta_{-s}Z' \parallel^2 + \frac{\parallel \mathcal{F}\sqrt{|v_z|}\beta r \parallel^2}{\delta_1|\xi|^2} + \parallel (I-P)\Phi \parallel^2 \Big) \\ &-2\sum_{j=0}^{4}\int \bar{\psi}_j\chi^c\chi_1^c\hat{U}\Big(\mathcal{F}\hat{\varphi}\frac{\partial\beta}{\partial\bar{\tau}}\Big)Mdv\Big(\int \bar{\psi}_j\chi^c\chi_1^c(\mathcal{F}\hat{\varphi}\beta-\hat{U}Z)Mdv\Big)^*.\end{aligned}$$

Therefore for $\xi \neq (0,0)$,

$$\begin{aligned}&\int (P\Phi)^2(\sigma,\xi,v)Mdvd\sigma \\ &\le C\Big(\frac{1}{\varepsilon^2}\int d\sigma \parallel \zeta_{-s}(v)\mathcal{F}\widehat{L_J^*\varphi})(\sigma,\xi,\cdot)\|^2 + \int d\sigma\Big(\parallel (I-P)\Phi(\sigma,\xi,\cdot) \parallel^2 \\ &+\frac{\parallel \mathcal{F}\sqrt{|v_z|}\beta r \parallel^2}{\delta_1|\xi|^2}\Big) + \int \parallel \nu^{-\frac{1}{2}}\hat{g}\beta(\bar{\tau},\xi,\cdot) \parallel^2 d\bar{\tau}\Big) \\ &-2\sum_{j=0}^{4}\int d\sigma\int \bar{\psi}_j\chi^c\chi_1^c\hat{U}(\mathcal{F}\hat{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv\Big(\int \bar{\psi}_j\chi^c\chi_1^c(\mathcal{F}\hat{\varphi}\beta-\hat{U}Z)Mdv\Big)^*.\end{aligned}$$

Sending τ_0 to zero implies that

$$\begin{aligned}\int_0^\infty d\bar{\tau}\int (P\hat{\varphi})^2(\bar{\tau},\xi,v)Mdv &\le C\Big(\frac{1}{\varepsilon^2}\int_0^\infty d\bar{\tau}\Big(\parallel \zeta_{-s}(v)\widehat{L_J^*\varphi})(\bar{\tau},\xi,\cdot) \parallel^2 \\ &\quad + \parallel (I-P)\hat{\varphi}(\bar{\tau},\xi,\cdot) \parallel^2 \Big) + \int_0^\infty d\bar{\tau}\int \nu^{-1}\hat{g}^2(\bar{\tau},\xi,v)Mdv \\ &\quad + \int_0^\infty d\bar{\tau}\frac{\parallel \sqrt{|v_z|}r \parallel^2}{\delta_1|\xi|^2}\Big).\end{aligned} \tag{4.3}$$

Taking δ and ε small enough and summing the previous inequality over all $0 \neq \xi \in \mathbb{Z}^2$, implies, by the Parseval inequality, that

$$\begin{aligned}&\int_0^\infty\int (\widetilde{P\varphi})^2(\bar{\tau},x,z,v)Mdvdxdzd\bar{\tau} \\ &\le c\Big(\frac{1}{\varepsilon^2}\int_0^\infty\int \nu((I-P)\varphi)^2(\bar{\tau},x,z,v)Mdvdxdzd\bar{\tau} \\ &+\int_0^\infty\int \nu^{-1}g^2(\bar{\tau},x,z,v)Mdvdxdzd\bar{\tau} + \parallel \gamma^-\varphi \parallel^2_{2\infty,2,\sim} + \eta \parallel \varphi \parallel^2_{2,2,2}\Big).\end{aligned}$$

As in Section 2, to use an argument based on a variant of Green's formula, we multiply the equation (4.2) by $2\varphi M\kappa$, and integrate over $[0,\bar T]\times[0,2\pi]^2\times\mathbb{R}^3$, integrate by parts and obtain, by using the spectral inequality (2.8) and the bounds $1\le\kappa(z)\le \mathrm{e}^{2\varepsilon G\pi}$,

$$\begin{aligned}&\parallel \gamma^-\varphi\parallel^2_{2\bar T,2,\sim}+\parallel\varphi\parallel^2_{2\bar T,2,2}+\frac{1}{\varepsilon}\parallel \nu^{\frac12}(I-P)\varphi\parallel^2_{2\bar T,2,2}\\&\le c(\varepsilon\parallel \nu^{-\frac12}(I-P)g\parallel^2_{2\bar T,2,2}+\eta_1\parallel P\varphi\parallel^2_{2\bar T,2,2}+\frac{1}{\eta_1}\parallel Pg\parallel^2_{2\bar T,2,2}).\end{aligned}$$

Inserting this into the previous inequality, the lemma follows. □

We next decompose the operator H in the remainder equation in accordance with the operator L_J: $H(\,\cdot\,)=\tilde J(q,P\,\cdot\,)+H_1(\,\cdot\,)$. We notice that $H_1(R)$ is of order zero in ε, and only depends on the nonhydrodynamic part $(I-P)R$. As in Section 2, to solve the equation for R we shall use an iteration procedure based on the decomposition of R in the sum R_1+R_2, where R_1 and R_2 are solutions of two different problems. R_1 solves

$$\begin{aligned}&\frac{\partial R_1}{\partial t}+\frac{1}{\varepsilon}v^\mu\nabla\cdot R_1+\frac{1}{\varepsilon}v_z\cdot\frac{\partial R_1}{\partial z}-\frac{G}{M}\frac{\partial(MR_1)}{\partial v_z}=\frac{1}{\varepsilon^2}L_JR_1\\&\qquad\qquad\qquad\qquad+\frac{1}{\varepsilon}H_1(R_1)+\frac{1}{\varepsilon}g,\\&R_1(0,x,z,v)=R_0(x,z,v),\\&R_1(t,x,\mp\pi,v)=-\frac{1}{\varepsilon}\bar\psi(t,x,\mp\pi,v),\quad t>0,\ v_z\gtrless 0,\end{aligned}\tag{4.4}$$

Here R_1 is periodic in x of period 2π, and g is some given function, x-periodic of period 2π with $\int Mg(\cdot,x,z,v)dxdzdv\equiv 0$. For the existence of the solution, see the discussion of (4.2). The equation for R_2 will be introduced below in (4.7).

The non-hydrodynamic part of R_1 is again estimated by Green's formula; multiply (4.4) by $2R_1M\kappa$, integrate with respect to the variables $(\bar\tau,x,z,v)$ over $[0,\bar T]\times[0,2\pi]^2\times\mathbb{R}^3$, integrate by parts and use the spectral inequality for L_J and the bounds $1\le\kappa(z)\le\mathrm{e}^{2\varepsilon G\pi}$, to obtain, for every $\eta_1>0$,

$$\begin{aligned}\parallel\gamma^-R_1\parallel^2_{2\bar T,2,\sim}&+\parallel R_1(\bar T)\parallel^2_{2,2}+\frac{1}{\varepsilon}\parallel\nu^{\frac12}(I-P_J)R_1\parallel^2_{2\bar T,2,2}\\&\le c\Big(\parallel R_0\parallel^2_{2,2}+\varepsilon\parallel\nu^{-\frac12}(I-P_J)g\parallel^2_{2\bar T,2,2}\\&+\frac{\eta_1}{2}\parallel P_JR_1\parallel^2_{2\bar T,2,2}+\frac{1}{2\eta_1}\parallel P_Jg\parallel^2_{2\bar T,2,2}+\frac{1}{\varepsilon^2}\parallel\bar\psi\parallel^2_{2\bar T,2,\sim}\Big).\end{aligned}\tag{4.5}$$

An a priori bound for P_JR_1 is obtained in the following lemma based on dual techniques involving the problem (4.2). Consider first the problem (4.4) without the term $H_1(R_1)$.

Lemma 4.2 *Set $h := P_J R_1$. Then*

$$\begin{aligned}\| h \|^2_{2,2,2} &\le c(\| R_0 \|^2_{2,2} + \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2,2} \\ &+\frac{1}{\varepsilon^2} \| P_J g \|^2_{2,2,2} + \frac{1}{\varepsilon^3} \| \bar{\psi} \|^2_{2,2,\sim}).\end{aligned}$$

Proof of Lemma 4.2. In the variables $(\bar{\tau}, x, z, v)$, the function R_1 is 2π-periodic in x and solution to

$$\begin{aligned}&\frac{\partial R_1}{\partial \bar{\tau}} + \mu\ v_x \cdot \frac{\partial R_1}{\partial x} + v_z \cdot \frac{\partial R_1}{\partial z} - \varepsilon G M^{-1} \frac{\partial (M R_1)}{\partial v_z} = \frac{1}{\varepsilon} L_J R_1 + g, \quad (4.6)\\ &R_1(0,x,z,v) = R_0(x,z,v),\\ &R_1(\bar{\tau}, x, \mp\pi, v) = -\frac{1}{\varepsilon}\bar{\psi}(\bar{\tau}, x, \mp\pi, v), \quad \bar{\tau} > 0,\ v_z \gtrless 0,\end{aligned}$$

Let φ be a 2π-periodic function in x, solution to

$$\frac{\partial \varphi}{\partial \bar{\tau}} + \mu\ v_x \frac{\partial \varphi}{\partial x} + v_z \frac{\partial \varphi}{\partial z} - \varepsilon G M^{-1} \frac{\partial (M\varphi)}{\partial v_z} = \frac{1}{\varepsilon} L_J^* \varphi + h,$$

with zero initial values and ingoing boundary values at $z = -\pi, \pi$. Multiply the equation for φ by $\kappa M R_1$ and the one for R by $\kappa M \varphi$ sum them and integrate by parts. Then,

$$\begin{aligned}&\int d\bar{\tau} dx dz dv \left(M \frac{\partial}{\partial z}(v_z \kappa R_1 \varphi) - \varepsilon G \kappa \frac{\partial (M R_1 \varphi)}{\partial v_z} \right)\\ &+ \int d\bar{\tau} dx dz \kappa(z) \frac{\partial}{\partial \bar{\tau}} (R_1, \varphi)_H\\ &= \int d\bar{\tau} dx dz M \kappa dv [\frac{1}{\varepsilon}(L_J((I-P_J)R_1)(I-P)\varphi)\\ &+ \frac{1}{\varepsilon}((I-P_J)(R_1) L_J^*(I-P)\varphi) + g\varphi + h P_J R_1].\end{aligned}$$

This gives

$$\begin{aligned}\| h \|^2_{2\bar{\tau},2,2} \le\ & \frac{K_1}{2} \| R_1(\bar{\tau},\cdot,\cdot) \|^2_{2,2} + \frac{1}{2K_1} \| \varphi(\bar{\tau},\cdot,\cdot) \|^2_{2,2}\\ &+ \frac{K_1}{2} \| \gamma^- R_1 \|^2_{2\bar{\tau},2,\sim} + \frac{1}{2K_1} \| \gamma^- \varphi \|^2_{2\bar{\tau},2,\sim}\\ &+ \frac{K_3}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P_J)R_1 \|^2_{2\bar{\tau},2,2} + \frac{1}{K_3 \varepsilon} \| \nu^{\frac{1}{2}}(I-P)\varphi \|^2_{2\bar{\tau},2,2}\\ &+ \frac{K_4}{2} \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2\bar{\tau},2,2} + \frac{1}{2K_4} \| \nu^{\frac{1}{2}}(I-P)\varphi \|^2_{2\bar{\tau},2,2}\\ &+ \frac{\varepsilon^2}{2K_4} \| \nu^{\frac{1}{2}}\varphi \|^2_{2,2} + \frac{K_2}{2} \| P_J g \|^2_{2\bar{\tau},2,2} + \frac{1}{2K_2} \| P\varphi \|^2_{2\bar{\tau},2,2}\\ &+ \frac{\varepsilon^2}{2K_2} \| \nu^{\frac{1}{2}}\varphi \|^2_{2,2},\end{aligned}$$

for any positive constants K_j, $j = 1, ..., 4$. All the terms computed at time $\bar\tau$ on the l.h.s can be estimate using Lemma 4.1 and (4.5), leading to

$$\begin{aligned}\| h \|_{2,2,2}^2 \leq c[(K_1+K_3) \| R_0 \|_{2,2}^2 \\ +(\frac{K_1}{\varepsilon^2}+\frac{K_3}{\varepsilon^2}) \| \bar\psi \|_{2,2,\sim}^2 +(\varepsilon K_1 + K_4 + K_3\varepsilon) \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2,2}^2 \\ +(\frac{1}{\varepsilon K_1}+\frac{1}{\varepsilon^2 K_2}+\frac{1}{\varepsilon K_3}+\frac{1}{K_4}+\frac{\eta_1}{\varepsilon^2 K_1}) \| h \|_{2,2,2}^2 \\ +(\frac{K_1}{\eta_1}+\frac{K_3}{\eta_1}+K_2) \| P_J g \|_{2,2,2}^2 + \| P_J R_1 \|_{2,2,2}^2 (\eta_1 K_1 + \eta_1 K_3) \\ +\eta(\frac{\varepsilon}{K_1}+\frac{\eta_1}{K_1}+\frac{\varepsilon}{K_3}+\frac{\varepsilon^2}{K_4}+\frac{1}{K_2}) \|< P_{J^*}\varphi >\|_{2,2}^2].\end{aligned}$$

We are left with the term $< P_{J^*}\varphi >$. As in Section 2, we use an approach based on ordinary differential equations for the Fourier transform with respect to the time and x-variables. Namely, the quantity $< \varphi >:= (2\pi)^{-1}\int \varphi(\cdot, x, \cdot)dx$ satisfies a 1-d problem including a small perturbation of magnitude δ from the value at the bifurcation point. After a Fourier transform in time (Fourier variable σ) the case of $|\varepsilon\sigma| < \sigma_0$ with $0 < \sigma_0$ sufficiently small, can be handled as in Lemma 2.2. For the remaining σ's use the term $i\varepsilon\sigma\mathcal{F}_{\bar\tau}\mathcal{F}_x\beta\varphi(\sigma, \xi_x, z)$ to express the ψ_0-moment. With $c_2 = (v_z^2, \psi_4)(v_z^2\bar A, \psi_4)^{-1}$, project the equation along $v_z - c_2 v_z \bar A$, and along ψ_0 and use the equation, leading to an expression for

$$\mathcal{F}_{\bar\tau}\mathcal{F}_x(\varepsilon\frac{\partial}{\partial t}\beta(\varphi_{v_z} - c_2\varphi_{v_z\bar A}) + \frac{\partial}{\partial z}(\beta\varphi_0 + \vartheta_2))$$

and

$$\mathcal{F}_{\bar\tau}\mathcal{F}_x(\varepsilon\frac{\partial}{\partial t}(\beta\varphi_0) + \frac{\partial}{\partial z}(\beta\varphi_{v_z})),$$

with ϑ_2 a nonhydrodynamic moment of φ, thus to an expression for

$$-i\varepsilon\sigma(|\mathcal{F}_{\bar\tau}\mathcal{F}_x(\beta\varphi_0)|^2+|\mathcal{F}_{\bar\tau}\mathcal{F}_x(\beta\varphi_{v_z})|^2)+\frac{\partial}{\partial z}((\mathcal{F}_{\bar\tau}\mathcal{F}_x(\beta\varphi_0+\zeta_2))(\mathcal{F}_{\bar\tau}\mathcal{F}_x(\beta\varphi_{v_z}))^*).$$

(For details cfr Lemma 5.5 below). Estimating the ingoing terms, this results in

$$\begin{aligned}\|< P\varphi >\|_{2,2} &\leq c \|< P\varphi >_x\|_{2,2,2} \leq c(\|< P\varphi >_x\|_{2,2,2} +\varepsilon \| \nu^{\frac{1}{2}}\varphi \|_{2,2,2}) \\ &\leq \frac{c}{\varepsilon} \|< h >_x\|_{2,2,2} +\eta \| \nu^{\frac{1}{2}}\varphi \|_{2,2,2} \leq \frac{c}{\varepsilon} \| h \|_{2,2,2} +\eta \| \nu^{\frac{1}{2}}\varphi \|_{2,2,2} .\end{aligned}$$

So choosing $\varepsilon < 1$, then K_1 and K_3 (resp.K_2) of order ε^{-1} (resp. ε^{-2}) and η_1 of order ε, leads to

$$\begin{aligned}\| h \|_{2,2,2}^2 &\leq c\Big(\frac{1}{\varepsilon} \| R_0 \|_{2,2}^2 + \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2,2}^2 +\frac{1}{\varepsilon^2} \| P_J g \|_{2,2,2}^2 \\ &+\frac{1}{\varepsilon^3} \| \bar\psi \|_{2,2,\sim}^2 +\eta \|< P_J R_1 >\|_{2,2,2}^2 \Big).\end{aligned}$$

This ends the proof of Lemma 4.2, when coming back to the t-variable. □

We now give the final estimates for R_1.

Lemma 4.3 *The solution R_1 to (4.4) satisfies*

$$\begin{aligned}
\| \nu^{\frac{1}{2}} R_1 \|_{2,2,2} &\leq c\Big(\| R_0 \|_{2,2} + \| \sqrt{\nu}(I-P_J)g \|_{2,2,2} \\
&\quad + \frac{1}{\varepsilon} \| P_J g \|_{2,2,2} + \frac{1}{\varepsilon^{\frac{3}{2}}} \| \bar{\psi} \|_{2,2,\sim} \Big), \\
\| R_1 \|_{\infty,2,2} &\leq c\Big(\| R_0 \|_{2,2} + \| \sqrt{\nu}(I-P_J)g \|_{2,2,2} + \frac{1}{\varepsilon} \| P_J g \|_{2,2,2} \\
&\quad + \frac{1}{\varepsilon^{\frac{3}{2}}} \| \bar{\psi} \|_{2,2,\sim} \Big), \\
\| \nu^{\frac{1}{2}} R_1 \|_{\infty,\infty,2} &\leq c\Big(\varepsilon^{-1} \| R_0 \|_{2,2} + \| R_0 \|_{\infty,2} + \frac{1}{\varepsilon} \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2,2} \\
&\quad + \frac{1}{\varepsilon^2} \| P_J g \|_{2,2,2} + \varepsilon \| \nu^{-\frac{1}{2}} g \|_{\infty,\infty,2} + \varepsilon^{-\frac{5}{2}} \| \bar{\psi} \|_{2,2,\sim} \\
&\quad + \frac{1}{\varepsilon} \| \bar{\psi} \|_{\infty,2,\sim} \Big).
\end{aligned}$$

<u>Proof of Lemma 4.3.</u> The solution R_1 of (4.4) without H_1-term satisfies

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}} \| \gamma^- R_1 \|_{2,2\sim} + \sup_{t\geq 0} \| R_1(t) \|_{2,2} + \frac{1}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P_J)R_1 \|_{2,2,2} \\
&\leq c\Big(\| R_0 \|_{2,2} + \frac{1}{\varepsilon^{\frac{3}{2}}} \| \bar{\psi} \|_{2,2,\sim} + \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2,2} \\
&+ \frac{\eta}{\sqrt{\varepsilon}} \| P_J R_1 \|_{2,2,2} + \frac{1}{\eta\sqrt{\varepsilon}} \| P_J g \|_{2,2,2} \Big),
\end{aligned}$$

for any $\eta > 0$. Moreover, it follows from Lemma 4.2 that

$$\begin{aligned}
\| P_J R_1 \|_{2,2,2} &\leq c\Big(\| R_0 \|_{2,2} + \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2,2} + \frac{1}{\varepsilon} \| P_J g \|_{2,2,2} \\
&\quad + \frac{1}{\varepsilon\sqrt{\varepsilon}} \| \bar{\psi} \|_{2,2,\sim} \Big).
\end{aligned}$$

Choosing $\eta = \sqrt{\varepsilon}$ leads to the first two inequalities of Lemma 4.3. Then, to get the L^∞ estimates, one has to study the solution along the characteristics. This analysis is complicated by the presence of the force, but can be done along the lines in [1] and the result is

$$\begin{aligned}
\| \nu^{\frac{1}{2}} R_1 \|_{\infty,\infty,2} &\leq c\Big(\frac{1}{\varepsilon} \| R_1 \|_{\infty,2,2} + \| R(0,\cdot) \|_{\infty,2} + \varepsilon \| \nu^{-\frac{1}{2}} g \|_{\infty,\infty,2} \\
&\quad + \| \gamma^+ R_1 \|_{\infty,2,\sim} \Big),
\end{aligned}$$

which leads to the last inequality of Lemma 4.3. Adding the term $\varepsilon^{-1} H_1(R_1)$ does not change these results. □

The remaining part R_2 of R satisfies the equation

$$\varepsilon\frac{\partial R_2}{\partial t}+\mu v_x\frac{\partial R_2}{\partial x}+v_z\frac{\partial R_2}{\partial z}-\varepsilon\frac{G}{M}\frac{\partial(MR_2)}{\partial v_z}=\frac{1}{\varepsilon}L_JR_2+H_1(R_2), \quad (4.7)$$
$$R_2(0,x,z,v)=0,$$
$$R_2(t,x,\mp\pi,v)=\frac{M_\mp(v)}{M(v)}\int_{w_z\lesseqgtr 0}\Big(R_1(t,x,\mp\pi,w)+R_2(t,x,\mp\pi,w)$$
$$+\frac{1}{\varepsilon}\bar\psi(t,x,\mp\pi,w)\Big)|w_z|Mdw,\quad t>0,\ v_z\gtrless 0,$$

Its analysis is more involved and requires a careful study of the Fourier transform of R_2. As with the stationary case in Section 2, existence for the problem (4.7) can be adapted from the corresponding study in [Ma], if one includes into that approach the spectral estimate for L_J, and the characteristics due to the force term.

In (4.7) the given indata part is

$$f^-(t,x,\mp\pi,v)\ =\frac{M_\mp}{M}\int_{w_z\lesseqgtr 0}\Big(R_1(t,x,\mp\pi,w)+\frac{1}{\varepsilon}\bar\psi(t,x,\mp\pi,w)\Big)|w_z|Mdw,$$
$$v_z\gtrless 0,$$

By Green's formula for (4.7), and noting that $H_1(R_2)$ only depends on $(I-P_0)R_2$, we get

$$\varepsilon\|R_2(t)\|^2_{2,2}+\parallel\gamma^-R_2\parallel^2_{2t,2,\sim}+\frac{c}{\varepsilon}\parallel\nu^{\frac12}(I-P_J)R_2\parallel^2_{2t,2,2}\leq\parallel\gamma^+R_2\parallel^2_{2t,2,\sim}\ . \quad (4.8)$$

This estimate is not yet final. In fact, compared with the analogous estimate for φ and R_1, the boundary terms here are different, due to the diffusive boundary conditions for R_2. We follow the reasoning in the stationary case for (2.28) - (2.29). Taking into account the differences, like dependence on time, we get

$$\varepsilon\|R_2\|^2_{2,2}(t)\ +\frac{c}{\varepsilon}\parallel\nu^{\frac12}(I-P_J)R_2\parallel^2_{2t,2,2} \quad (4.9)$$
$$\leq\frac{1}{\varepsilon\eta}\parallel f^-\parallel^2_{2t,2,\sim}+C\varepsilon\eta\parallel P_JR_2\parallel^2_{2t,2,2},$$

$$\parallel\gamma^-R_2\parallel^2_{2t,2,\sim}\leq\frac{1}{\varepsilon^2}\parallel f^-\parallel^2_{2t,2,\sim}+C\parallel P_JR_2\parallel^2_{2t,2,2}\ . \quad (4.10)$$

The hydrodynamic estimates for R_2 are obtained similarly to the stationary case. We again start with the 1-d (x-independent) case, with an inhomogeneous term g_1 which will take into account the x-dependence in later proofs. Reduce the equation (4.7) to a 1-d problem for $\int dxR_2(t,x,z,v):=R_2(t,z,v)$, with an inhomogeneous term g_1 such that $g_{10}=\int_{\mathbb{R}^3}dvg_1=0$:

$$\varepsilon\frac{\partial R_2}{\partial t}+v_z\frac{\partial R_2}{\partial z}-\varepsilon GM^{-1}\frac{\partial(MR_2)}{\partial v_z}=\frac{1}{\varepsilon}L_JR_2+H_1(R_2)+g_1. \quad (4.11)$$

Then, for the solution of this new problem the following lemma holds:

Lemma 4.4

$$\| P_J R_2 \|^2_{2,2,2} \leq \frac{c_1}{\varepsilon^2} \| f^- \|^2_{2,2\sim} + c_2(\| P_J R_1 \|^2_{2,2,2} + \| \nu^{-\frac{1}{2}} g_1 \|^2_{2,2,2}).$$

Proof of Lemma 4.4. The equation for the Fourier transform, $\hat{R}_2$ of $R_2(t,z,v)$ with respect to the space variable z is

$$\varepsilon \frac{\partial}{\partial t} \hat{R}_2 + i v_z \xi_z \hat{R}_2 - \varepsilon G M^{-1} \frac{\partial}{\partial v_z}(M \hat{R}_2) = \varepsilon^{-1} \widehat{L_J R_2} + \widehat{H_1(R_2)} - v_z r(-1)^{\xi_z} + \hat{g}_1, \tag{4.12}$$

$r(v)$ now denoting the difference between the ingoing and outgoing boundary values,

$$r(v) = R_2(t,\pi,v) - R_2(t,-\pi,v). \tag{4.13}$$

Starting from the method of Lemma 4.1 with L_J instead of its adjoint, and considering the ingoing boundary values as known, we reach (4.3) for $\xi_z \neq 0$ with obvious changes,

$$\int_0^\infty dt \int |P_{J0} \hat{R}_2|^2 (t,\xi_z,v) M dv \leq C\Big(\frac{1}{\varepsilon^2} \int_0^\infty dt \big(\| \zeta_{-s}(v) \widehat{L_J R_2}(t,\xi_z,\cdot) \|^2 + \| (I - P_{J0}) \hat{R}_2(t,\xi_z,\cdot) \|^2 \big) + \int_0^\infty dt \int \xi_z^{-2} \nu^{-1} |\hat{g_1}|^2 (t,\xi_z,v) M dv + \int_0^\infty dt \big(\frac{\| \sqrt{|v_z|} r \|^2}{\delta_1 |\xi_z|^2} + \varepsilon^2 \| \hat{R}_2 \|^2 \big)\Big).$$

Here, similarly to Section 2, we have fixed $z = z_0$, $t = t_0$ in the basis elements $\bar{\psi}_1$, ..., $\bar{\psi}_4$, writing P_{J0} for the corresponding kernel projection. At the end we replace P_{J0} with P_J, since $P_J - P_{J0} = O(\varepsilon)$. The sum $R = R_1 + R_2$ satisfies by hypothesis $\langle \hat{R}(0,\cdot) \rangle = \hat{R}_0(0) = 0$ for $t = 0$. The coefficients in the asymptotic expansion can be chosen so that, moreover, $\hat{R}_0(t,0) = 0$ for $t > 0$. That gives an estimate for $\hat{R}_{20}(0)$ in terms of $\hat{R}_{10}$.

For an estimate of the r-term, for $z = \pi$ we use (4.10) giving an estimate of the outflow at π in terms of known quantities and $P_{J0} R_2$. Then, for $\xi_z \geq \bar{\xi}_z$, and $\bar{\xi}_z$ large enough, the latter appears multiplied by a small factor and can be absorbed by the total sum on the left-hand side, taken over $\xi_z \geq \bar{\xi}_z$. Hence, for ξ large the bound stated in Lemma 4.4 is proved.

For the remaining bounded number of ξ's, each hydrodynamic mode is estimated separately. We obtain estimates for all the hydrodynamic moments in terms of $\hat{R}_{20}$, which will be the last one to be estimated.

For the v_z-moments multiply (4.12) by v_z (resp. one), integrate with respect to $M dv$, and multiply with $\hat{R}_{20}$ (resp. $\hat{R}_{2v_z}$). Use the notation $[f]_- = f(\xi_z) - (-1)^{\xi_z} f(0)$. Combining the results, it implies that for each fixed ξ_z

$$\varepsilon \frac{\partial}{\partial t} \left(\hat{R}_{20} [\hat{R}_{2v_z}]^*_- \right) = i \xi_z \hat{R}_{2v_z} [\hat{R}_{2v_z}]^*_- + i \xi \hat{R}^*_{2v_z^2} \hat{R}_{20} + \varepsilon G \int \Big\{ \hat{R}_{20}(\xi_z) v_z \frac{\partial}{\partial v_z} (M \hat{R}_2^*) \Big\} dv + [\hat{R}_{2v_z}]^*_- \int M \Big\{ v_z r(-1)^{\xi_z} + \hat{g}_1 \Big\} dv.$$

We integrate over $t \in [0, \bar{t}]$, to get an estimate for the term $i\xi_z|\hat{R}_{2v_z}(\xi_z)|^2$ on the right-hand side. The time derivative, then, produces a term $\varepsilon \hat{R}_{20}(\xi_z)[\hat{R}_{2v_z}]^*_-$ computed at time $\bar{t}$ which is bounded as

$$C\varepsilon(|\hat{R}_{20}(\xi_z)|^2 + |\hat{R}_{2v_z}(\xi_z)|^2 + |\hat{R}_{2v_z}(0)|^2) \leq \varepsilon \|PR_2\|^2_{2,2}(\bar{t}).$$

The latter is estimated by using Green's formula for (4.11), getting

$$\varepsilon \|PR_2\|^2_{2,2} \leq \frac{c}{\eta}(\| f^- \|^2_{2\bar{t},2,\sim} + \| \nu^{-\frac{1}{2}} g_1 \|^2_{2\bar{t},2,2}) + \eta \| P_J R_2 \|^2_{2\bar{t},2,2} .$$

The other terms can be easily estimated, but for the one containing the boundary r-term, which is equal to $\hat{R}_{2v_z}(\xi_z)(R_{2v_z}(t,\pi) - R_{2v_z}(t,-\pi))$. This term can be estimated by

$$\int \big(\eta |\hat{R}_{2v_z}|^2 + \frac{1}{\eta} \| f^- \|^2_{2\sim} \big) dt.$$

In conclusion, we have that for $\xi_z \neq 0$

$$\begin{aligned} \int |\hat{R}_{2v_z}(\xi_z)|^2 dt \ &\leq C \int dt \Big(\mid \hat{R}_{20}(\xi_z) \mid^2 + \mid \hat{R}_{2v_z}(0) \mid^2 \\ &+ \eta \| PR_2 \|^2_{2,2} + \frac{1}{\eta} \| f^- \|^2_{2\sim} + \| \nu^{-\frac{1}{2}} g_1 \|^2_{2,2} \Big). \end{aligned} \tag{4.14}$$

We then compute $\hat{R}_{2v_z}(t,0)$ for $\xi_z = 0$. Multiply (4.11) by M, integrate over velocity and over $z' \in [-\pi, z]$, followed by an integration over $[-\pi,\pi]$. We get, after multiplication by $\hat{R}_{2v_z}(t,0)$,

$$\begin{aligned} &\varepsilon \hat{R}_{2v_z}(t,0) \frac{\partial \overline{R_{20}}}{\partial t} + (\hat{R}_{2v_z}(t,0))^2 = \hat{R}_{2v_z}(t,0) \int_{-\pi}^{\pi} dz \int_{-\pi}^{z} dz' dv M g_1 \\ &\quad + 2\pi \hat{R}_{2v_z}(t,0) \int dv v_z M R_2(t,v,-\pi), \end{aligned} \tag{4.15}$$

where $\overline{R_{20}} := \int_{-\pi}^{\pi} dz \int_{-\pi}^{z} dz' dv M R_2$. Multiply (4.12) by Mv_z, integrate over velocity and multiply by $\overline{R_{20}}$,

$$\varepsilon \overline{R_{20}} \frac{\partial}{\partial t} \hat{R}_{2v_z} = \overline{R_{20}} \Big(r_{v_z^2} + \int v_z \hat{g}_1 M dv \Big) + \varepsilon G \overline{R_{20}} \int v_z \frac{\partial}{\partial v_z} M \hat{R}_2(0,v) dv.$$

Summing the last two equations,

$$\begin{aligned} &\varepsilon \frac{\partial}{\partial t}(\overline{R_{20}} \hat{R}_{2v_z}) + \hat{R}^2_{2v_z} = \overline{R_{20}} \Big(r_{v_z^2} + \int M v_z \hat{g}_1 dv \Big) \\ &+ \varepsilon G \overline{R_{20}} \int v_z \frac{\partial}{\partial v_z} M \hat{R}_2(0,v) dv \\ &+ \hat{R}_{2v_z} \int_{-\pi}^{\pi} dz \int_{-\pi}^{z} dz' dv M g_1 + \hat{R}_{2v_z} 2\pi \int dv v_z M R_2(t,v,-\pi) \end{aligned} \tag{4.16}$$

We use this relation to bound the time integral of $\hat{R}^2_{2v_z}$. We integrate over time, and use inequality (4.10) to control the boundary term $r_{v_z^2}$. The result is

$$\begin{aligned}\int |\hat{R}_{2v_z}(t,0)|^2 dt \ &\leq C\Big(\frac{1}{\eta}\int dt \|R_{20}\|_2^2 + \eta \parallel P_J R_2 \parallel^2_{2,2} \\ &+ \parallel \nu^{-\frac{1}{2}} g_1 \parallel^2_{2,2} + \parallel f^- \parallel^2_{2\sim}\Big).\end{aligned}$$

By Parseval identity, $\|R_{20}\|_2^2 = \sum_{\xi_z\neq 0} |\hat{R}_{20}(\xi_z)|^2 + |\hat{R}_{20}(0)|^2$. The last term is equal to $|\hat{R}_{10}(0)|^2$ because $\int dzdvR(t,z,v) = 0$. The sum for $\xi_z > \bar{\xi}_z$ was estimated above. In conclusion,

$$\begin{aligned}\int |\hat{R}_{2v_z}(0)|^2 dt \ &\leq C\Big(\sum_{0<\xi_z\leq\bar{\xi}_z} |\hat{R}_{20}(\xi_z)|^2 + \parallel P_J R_1 \parallel^2_{2,2,2} + \eta \parallel PR_2 \parallel^2_{2,2,2} \\ &+ \parallel \nu^{\frac{1}{2}} g_1 \parallel^2_{2,2,2} + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2,2} \\ &+ \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2,2} + \parallel f^- \parallel^2_{2,2\sim}\Big).\end{aligned} \tag{4.17}$$

For the control of v_x-moments, we use $\hat{R}_{2v_x}(\xi) = \hat{R}_{2v_z^2 v_x}(\xi) - \hat{R}^{\perp}_{2v_x v_z^2}(\xi)$ (recall that $\int v_z^2 v_x^2 M dv = 1$). Multiply (4.12) by $v_x v_z$ (resp. $v_x v_z^2$), integrate with respect to Mdv, and multiply with $\hat{R}_{2v_x v_z^2}$ (resp. $\hat{R}_{2v_x v_z}$). Adding the results implies for each ξ_z that

$$\begin{aligned}&\varepsilon\frac{\partial}{\partial t}(\hat{R}_{2v_x v_z}\hat{R}^*_{2v_x v_z^2}) = i\xi_z|\hat{R}_{2v_x v_z^2}|^2 + i\xi_z \hat{R}^*_{2v_x v_z^3}\hat{R}_{2v_x v_z} \\ &+\varepsilon G\int\Big(\hat{R}^*_{2v_x v_z^2} v_x v_z \frac{\partial M\hat{R}_2}{\partial v_z} + \hat{R}_{2v_x v_z} v_x v_z^2 \frac{\partial M\hat{R}^*_2}{\partial v_z}\Big)dv \\ &+\hat{R}^*_{2v_x v_z^2}\int M v_x v_z\Big(\frac{1}{\varepsilon}\widehat{L_J R_2} + \widehat{H_1(R_2)} + v_z r(-1)^{\xi_z} + \hat{g}_1\Big)dv \\ &+\hat{R}_{2v_x v_z}\int M v_x v_z^2\Big(\frac{1}{\varepsilon}\widehat{L_J R_2} + \widehat{H_1(R_2)} + v_z r(-1)^{\xi_z} + \hat{g}_1\Big)^* dv.\end{aligned}$$

We do not estimate directly the terms involving higher moments of the boundary term r. To remove these terms we subtract the same expression for $\xi_z = 0$ multiplied by $(-1)^{\xi_z}$, getting

$$\begin{aligned}&\varepsilon\frac{\partial}{\partial t}([\hat{R}_{2v_x v_z}]_-[\hat{R}_{2v_x v_z^2}]^*_-) = i\xi_z[\hat{R}_{2v_x v_z^2}]^*_-\hat{R}_{2v_x v_z^2}(\xi_z) + i\xi_z\hat{R}^*_{2v_x v_z^3}[\hat{R}_{2v_x v_z}]_- \\ &+\varepsilon G\int\Big\{[\hat{R}_{2v_x v_z^2}]^*_- v_x v_z\frac{\partial}{\partial v_z}(M[\hat{R}_2]_-)]) + [\hat{R}_{2v_x v_z}]_- v_x v_z^2\frac{\partial}{\partial v_z}(M[\hat{R}_2]_-)^*\Big\} dv \\ &+[\hat{R}_{2v_x v_z^2}]^*_-\int M v_x v_z\Big(\frac{1}{\varepsilon}[\widehat{L_J R_2}]_- + [\widehat{H_1(R_2)}]_- + [\hat{g}_1]_-\Big)dv \\ &+[\hat{R}_{2v_x v_z}]_-\int M v_x v_z^2\Big(\frac{1}{\varepsilon}[\widehat{L_J R_2}]_-] + [\widehat{H_1(R_2)}]_- + [\hat{g}_1]_-\Big)^* dv.\end{aligned}$$

Integrate with respect to t to obtain for $\xi \neq 0$

$$\int |\hat{R}_{2v_x}(\xi_z)|^2 dt \ \leq C \int \Big(\mid \hat{R}_{2v_x}(0) \mid^2 + \eta \parallel PR_2 \parallel^2_{2,2} + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2} + \parallel \nu^{-\frac{1}{2}} g_1 \parallel^2_{2,2} + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2} \Big) dt.$$

We have used that only $(I-P)R_2$ contributes to $\hat{R}_{2v_xv_z}$ and $\hat{R}_{2v_xv_z^3}$.

Now we discuss the estimate of $\hat{R}_{2v_x}$ for $\xi_z = 0$. To eliminate the outgoing boundary terms, we multiply (4.11) by Mv_xv_z and consider first the equation we get by taking the integral $\int_{-\pi}^{\pi} dz \int_{-\pi}^{z} dz' \int_{\mathbb{R}^3} dv$ and then the one we get by taking the integral $2\pi \int_{-\pi}^{\pi} dz \int_{v_z<0} dv$. By taking the difference of the two equations we get for the l.h.s. (but for the force terms)

$$\varepsilon \frac{\partial}{\partial t} \Big(\int dv \int_{-\pi}^{\pi} dz \int_{-\pi}^{z} dq v_x v_z M R_2(t,q,v) - 2\pi \int_{v_z<0} v_x v_z M \hat{R}_2(t,0,v) dv \Big) + \hat{R}_{2v_xv_z^2}(t,0)$$

The remaining ingoing boundary terms are zero. Now, notice that

$$\int_{v_z<0} v_x v_z M \hat{R}_2(t,0,v) = \zeta \hat{R}_{2v_x}(t,0) + \hat{R}_{2\perp}(t,0),$$

where $\zeta = \int_{v_z<0} v_x^2 v_z M dv$ and $\hat{R}_{2\perp}(t,0) = \int_{v_z<0} v_x v_z M \hat{R}_2^{\perp}(t,0,v) dv$ depends on the nonhydrodynamic part of $\hat{R}_2$. We multiply by $\hat{R}_{2v_x}(t,0)$ and, by using the equation for it, we get in the l.h.s (but for the force terms)

$$\varepsilon \frac{\partial}{\partial t} \Big(\mathcal{D} \hat{R}_{2v_x}(t,0) - 2\pi\varepsilon\zeta \hat{R}^2_{2v_x}(t,0) \Big) + \hat{R}_{2v_x} \hat{R}_{2v_xv_z^2} + \mathcal{D} \int dv v_x v_z (\gamma^- R_2(\pi) - \gamma^- R_2(-\pi))$$

where $\mathcal{D} : \int \int_{-\pi}^{\pi} \int_{-\pi}^{z} v_x v_z M R_2(t,q,v) dq dz dv - 2\pi \hat{R}_{2\perp}(t,0)$ is nonhydrodynamic. We use again $\hat{R}_{2v_x}(\xi) = \hat{R}_{2v_xv_z^2}(\xi) - \hat{R}^{\perp}_{2v_xv_z^2}(\xi)$ and estimate $\gamma^- R_2$ by (4.10). This gives

$$\int |\hat{R}_{2v_x}(0)|^2 dt \ \leq C \int dt \Big(\eta \parallel PR_2 \parallel^2_{2,2} + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2} + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2} + \parallel f^- \parallel^2_{2,\sim} + \parallel \nu^{-\frac{1}{2}} g_1 \parallel^2_{2,2} \Big).$$

The v_y-moments are analogous, hence for all ξ_z

$$\int |\hat{R}_{2v_x}(\xi_z)|^2 dt \ + \int |\hat{R}_{2v_y}(\xi_z)|^2 dt \leq c \int (\eta \parallel PR_2 \parallel^2_{2,2} + \parallel f^- \parallel^2_{2,\sim} + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2} + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2} + \parallel \nu^{-\frac{1}{2}} g_1 \parallel^2_{2,2}) dt. \tag{4.18}$$

Consider next the ψ_4-moment for $\xi_z \neq 0$. Multiply (4.12) by $v_z\bar{A}$ (resp. $v_z^2\bar{A}$), and integrate with respect to Mdv. Similarly to the proof of Lemma 2.4, this gives

$$\begin{aligned}\varepsilon\frac{\partial}{\partial t}(v_z\bar{A},\hat{R}_2(\xi_z)) &+(-1)^{\xi_z}r_{v_z^2\bar{A}}+i\xi_z(v_z^2\bar{A},\hat{R}_2(\xi_z))\\ &+\varepsilon G\int\frac{\partial}{\partial v_z}(v_z\bar{A}M)\hat{R}_2(\xi_z)dv=\frac{1}{\varepsilon}(v_z\bar{A},\widehat{L_JR_2}(\xi_z))\\ &+(v_z\bar{A},\widehat{H_1(R_2)}(\xi_z))+(v_z\bar{A},\hat{g}_1),\end{aligned}$$

$$\begin{aligned}\varepsilon\frac{\partial}{\partial t}(v_z^2\bar{A},\hat{R}_2(\xi_z)) &+(-1)^{\xi_z}r_{v_z^3\bar{A}}+i\xi_z(v_z^3\bar{A},\hat{R}_2(\xi_z))\\ &+\varepsilon G\int\frac{\partial}{\partial v_z}(v_z^2\bar{A}M)\hat{R}_2(\xi_z)dv=\frac{1}{\varepsilon}(v_z^2\bar{A},\widehat{L_JR_2}(\xi_z))\\ &+(v_z^2\bar{A},\widehat{H_1(R_2)}(\xi_z))+(v_z^2\bar{A},\hat{g}_1).\end{aligned}$$

Similarly to the v_x-case, we manipulate the equations to remove the boundary terms, leading to,

$$\begin{aligned}&\varepsilon\frac{\partial}{\partial t}([\hat{R}_{2v_z\bar{A}}]_-[\hat{R}_{2v_z^2\bar{A}}]^*_-)i\xi[\hat{R}_{2v_z^2\bar{A}}]^*_-\hat{R}_{2v_z^2\bar{A}}(\xi_z)-i\xi_z\hat{R}^*_{2v_z^3\bar{A}}(\xi_z)[\hat{R}_{2v_z\bar{A}}]_-\\ &+\varepsilon G\int dv\Big([\hat{R}_{2v_z^2\bar{A}}]^*_-\ v_z\bar{A}\frac{\partial}{\partial v_z}(M[\hat{R}_2]_-)+[\hat{R}_{2v_z\bar{A}}]_-\ v_z^2\bar{A}\frac{\partial}{\partial v_z}(M[\hat{R}_2]_-)\Big)\\ &+[\hat{R}_{2v_z^2\bar{A}}]^*_-\int dvMv_z\bar{A}\Big(\frac{1}{\varepsilon}[\widehat{L_JR_2}]_-+[\widehat{H_1(R_2)}]_-+[\hat{g}_1]_-\Big)\\ &+[\hat{R}_{2v_z\bar{A}}]_-\int dvMv_z^2\bar{A}\Big(\frac{1}{\varepsilon}[\widehat{L_JR_2}]_-+[\widehat{H_1(R_2)}]_-+[\hat{g}_1]_-\Big)^*.\end{aligned}$$

It follows that for $\xi_z \neq 0$

$$\begin{aligned}\int dt\mid\hat{R}_{24}(\xi)\mid^2\ &\leq C\int dt\Big(\mid\hat{R}_{24}(0)\mid^2+\eta\mid\hat{R}_{20}(\xi_z)\mid^2+\eta\mid\hat{R}_{2v_z}(0)\mid^2\\ &+\eta\parallel PR_2\parallel^2_{2,2}+\parallel(I-P)R_2\parallel^2_{2,2}+\frac{1}{\varepsilon^2}\parallel\nu^{\frac{1}{2}}(I-P_J)R_2\parallel^2_{2,2}\\ &+\parallel f^-\parallel^2_{2,\sim}+\parallel\nu^{-\frac{1}{2}}g_1\parallel^2_{2,2}\Big).\end{aligned}\tag{4.19}$$

For $\xi_z = 0$ multiplying (4.11) with $v_z\bar{A}$ and arguing similarly to the proof of (4.18), gives

$$\begin{aligned}\int|\hat{R}_{24}(0)|^2dt\leq c\int dt\Big(\eta\parallel PR_2\parallel^2_{2,2}+\parallel\nu^{\frac{1}{2}}(I-P)R_2\parallel^2_{2,2}\\ +\frac{1}{\varepsilon^2}\parallel\nu^{\frac{1}{2}}(I-P_J)R_2\parallel^2_{2,2}+\parallel f^-\parallel^2_{2,\sim}+\parallel\nu^{-\frac{1}{2}}g_1\parallel^2_{2,2}\Big).\end{aligned}\tag{4.20}$$

The only moments still to be estimated are the ψ_0-moments for $\xi_z \neq 0$.

With $c_2 = (v_z^2, \psi_4)(v_z^2 \bar{A}, \psi_4)^{-1}$ and $c_3 = (v_z^4, 1)(v_z^6, 1)^{-1}$, proceed similarly to the corresponding ψ_4-case discussed earlier, but start from $v_z - c_2 v_z \bar{A}$ instead of $v_z \bar{A}$, and $v_z^2 - c_3 v_z^4$ instead of $v_z^2 \bar{A}$. That gives

$$\int dt \mid \hat{R}_{20}^2(\xi_z) \mid \leq C \int dt \Big(\parallel P_J R_1 \parallel_{2,2}^2 + \eta \parallel P R_2 \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}} g_1 \parallel_{2,2}^2 + \parallel (I-P) R_2 \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}} (I - P_J) R_2 \parallel_{2,2}^2 + \parallel f^- \parallel_{2,\sim}^2 \Big). \quad (4.21)$$

The lemma is proved by collecting the estimates above. □

By using this 1-d analysis, it follows in the 2-d case that

Lemma 4.5 *The solution R_2 of (4.7) satisfies*

$$\parallel P_J R_2 \parallel_{2,2,2}^2 \leq c\Big(\frac{1}{\varepsilon^2} \parallel f^- \parallel_{2,2\sim}^2 + \parallel P_J R_1 \parallel_{2,2,2}^2\Big).$$

Proof of Lemma 4.5 We apply Lemma 4.4 to $\hat{R}_2(0, \xi_z, v) = \int dx R_2(x, z, v)$ and, taking into account that g_1 is of order δ, get a bound for the Fourier components $P_J \hat{R}_2(0, \xi_z)$, for δ small. As discussed at the beginning of the proof of Lemma 4.4, the components with ξ large are under control, since the r-terms are small after division by $|\xi|^2$. The remaining components in the case $\xi_x \neq 0$, ξ_x, ξ_z finite, are estimated by analyzing the equations for the moments of R_2 and applying in a suitable way Lemma 4.4. The proof follows closely the one of Lemma 4.4, so we will not give all details but only point out the differences.

Consider first the v_x-moment for $\xi_z = 0$. Multiply the (spatial) Fourier version of (4.7) by Mdv (resp. $v_x M dv$) and integrate. Combining the results

$$\varepsilon \frac{\partial}{\partial t} (\hat{R}_{20}(\xi_x, 0) \hat{R}_{2v_x}^*(\xi_x, 0)) = i\mu \xi_x \hat{R}_{2v_x}(\xi_x, 0) \hat{R}_{2v_x}^*(\xi_x, 0) + i\mu \xi_x \hat{R}_{2v_x^2}^*(\xi_x, 0) \hat{R}_{20}(\xi_x, 0) + \hat{R}_{2v_x}^*(\xi_x, 0) \int dv M v_z r + \hat{R}_{20}(\xi_x, 0) \int dv M v_x v_z r^*.$$

This gives

$$\int |\hat{R}_{2v_x}|^2(\xi_x, 0) dt \leq C \int dt \Big(|\hat{R}_{20}|^2(\xi_x, 0) + \eta \parallel P R_2 \parallel_{2,2}^2 + \parallel \nu^{\frac{1}{2}} (I-P) R_2 \parallel_{2,2}^2 + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}} (I - P_J) R_2 \parallel_{2,2}^2 + \parallel f^- \parallel_{2\sim}^2 \Big). \quad (4.22)$$

To bound the v_z-moments for $\xi_z \neq 0$, we use a variant of the proof in Lemma 4.4. Multiply the Fourier version of (4.7) by $((v_x^2, v_z^2 \bar{B}) - v_z^2 \bar{B}) M dv$ and integrate. That removes the hydrodynamic ξ_x-term. To remove the $v_z r$-term we also subtract the same expression for $\xi_z = 0$ multiplied with $(-1)^{\xi_z}$. We use the notation $[f]^x := f(\xi) - (-1)^{\xi_z} f(\xi_x, 0)$. Proceeding as in Lemma 4.4 gives

$$\varepsilon \frac{\partial}{\partial t}([\hat{R}_{20}(v_x^2, v_z^2\bar{B}) - \hat{R}_{v_z^2\bar{B}})]^x[\hat{R}^*_{2v_z}]^x) = i\xi_z(\hat{R}_{2v_z}(v_x^2, v_z^2\bar{B}) - \hat{R}_{2v_z^3\bar{B}})[\hat{R}^*_{2v_z}]^x$$
$$+ i\mu\xi_x\Big([\hat{R}_{v_x}(v_x^2, v_z^2\bar{B}) - \hat{R}_{v_xv_z^2\bar{B}}]^x + [\hat{R}^*_{2v_xv_z}]^x\Big)[\hat{R}^*_{2v_z}]^x$$
$$+ \xi_z\hat{R}^*_{2v_z^2}(\xi))[\hat{R}_{20}(v_x^2, v_z^2\bar{B}) - \hat{R}_{v_z^2\bar{B}}]^x - \varepsilon G[\hat{R}^*_{2v_z}]^x \int v_z^2\bar{B}\frac{\partial}{\partial v_z}(M[\hat{R}_2]^x)dv$$
$$+ \varepsilon G[\hat{R}_{20}(\xi) - \hat{R}_{v_z^2\bar{B}}]^x \int v_z \frac{\partial}{\partial v_z}(M[\hat{R}^*_2]^x dv)$$
$$+ [\hat{R}^*_2]^x \int M\Big(\frac{1}{\varepsilon}v_z^2\bar{B}[\widehat{L_JR_2}]^x - v_z^2\bar{B}[\widehat{H_1(R_2)}]^x\Big)dv.$$

It follows after integration with respect to t that for $\xi_z \neq 0$

$$\int |\hat{R}_{2v_z}|^2(\xi)dt \ \leq C\int dt\Big(|\hat{R}_{2v_z}|^2(\xi_x, 0) + |\hat{R}_{20}|^2(\xi) + \eta \parallel PR_2 \parallel^2_{2,2} \tag{4.23}$$
$$+ \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2} + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2}\Big).$$

The moment $\hat{R}_{2v_x}(\xi_x, 0)$ is under control by (4.22), and so the previous approach for the v_x-moment when $\xi_z \neq 0$, gives

$$\int |\hat{R}_{2v_x}|^2(\xi)dt \leq C\int dt\Big(|\hat{R}_{20}|^2(\xi) + |\hat{R}_{24}|^2(\xi) + \eta \parallel PR_2 \parallel^2_{2,2}$$
$$+ \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2} + \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2} + \parallel f^- \parallel^2_{2\sim}\Big). \tag{4.24}$$

The $\hat{R}_{2v_y}(\xi)$-moment for $\xi_z \neq 0$ is similarly treated, starting from

$$\varepsilon\frac{\partial}{\partial t}\Big\{(\hat{R}_{2v_yv_z}(\xi) - (-1)^{\xi_z}\hat{R}_{2v_yv_z}(\xi_x, 0))(\hat{R}_{2v_yv_z^2}(\xi) - \hat{R}_{2v_yv_z^2}(\xi_x, 0)\Big\}.$$

Inserting the corresponding right-hand sides and estimating the upcoming moments, gives for $\xi_z \neq 0$

$$\int |\hat{R}_{2v_y}(\xi)|^2 dt \ \leq C\int dt\Big(\mid \hat{R}_{2v_y}(0) \mid^2 + \eta \parallel PR_2 \parallel^2_{2,2} + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel^2_{2,2}$$
$$+ \frac{1}{\varepsilon^2} \parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel^2_{2,2}\Big). \tag{4.25}$$

For the $\hat{R}_{2v_z}(\xi_x, 0)$-moment, use the procedure of Lemma 4.4 with the term $i\mu\xi_x\mathcal{F}_xR_{2v_x}(\xi_x, \cdot)$ added as an inhomogeneous term. Thus

$$\varepsilon\frac{\partial}{\partial t}\int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq\mathcal{F}_xR_{20}(\xi_x, q) \ = i\mu\xi_x\int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq\mathcal{F}_xR_{2v_x}(\xi_x, q)$$
$$+ \hat{R}_{2v_z}(\xi_x, 0) - 2\pi\int_{v_z>0} Mv_z\mathcal{F}_xf^-(\xi_x, -\pi)dv.$$

The equation for $\hat{R}_{2v_z}$ is

$$\varepsilon\frac{\partial}{\partial t}\hat{R}_{2v_z} = i\mu\xi_x\hat{R}_{2v_xv_z}(\xi_x,0) + r_{v_z^2}(\xi_x) + \varepsilon G\int v_z\frac{\partial}{\partial v_z}M\hat{R}_2(\xi_x,0,v)dv.$$

The resulting terms are of the same type we get before, except for

$$\int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq\mathcal{F}_xR_{2v_x}(\xi_x,q) \qquad \text{and} \qquad \int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq\mathcal{F}_xR_{20}(\xi_x,q)\ .$$

The former can be controlled using the observation $\| \mathcal{F}_xR_{2v_x}(\xi_x,.) \|_2^2 = \sum_{\xi_z}|\hat{R}_{2v_x}(\xi_x,\xi_z)|^2$, together with the estimates (4.22), (4.24) for the right-hand side. We conclude:

$$\int|\hat{R}_{2v_z}|^2(\xi_x,0)dt \le C\int dt\Big(\| R_{20} \|_{2,2}^2 + \eta \| PR_2 \|_{2,2}^2 + \| \nu^{\frac12}(I-P)R_2 \|_{2,2}^2 + \frac{1}{\varepsilon^2} \| \nu^{\frac12}(I-P_J)R_2 \|_{2,2}^2 + \| f^- \|_{2\sim}^2 \Big). \tag{4.26}$$

For the moment $\hat{R}_{2v_y}(\xi_x,0)$ we follow the corresponding procedure of Lemma 4.4. The new terms which result from the x-derivative, $\hat{R}_{2v_xv_y}(\xi_x,0)$ and $\hat{R}_{2v_xv_yv_z}(\xi_x,0)$, are nonhydrodynamic, and so the estimate (4.18) also holds for this moment.

The ψ_4-moments for $\xi_z \neq 0$ are treated as in the 1-d case. The extra terms resulting from the x-derivative, contain at least one nonhydrodynamic factor. The resulting inequality is again (4.19).

The moment $\hat{R}_{24}(\xi_x,0)$ is obtained as in the 1-d case, but here with an additional term
$\int|\hat{R}_{20}|^2(\xi_x,0)dt$ to the right in the final estimate (4.20). Thus

$$\int|\hat{R}_{24}|^2(\xi_x,0,t)dt \le C\int dt\Big(|\hat{R}_{20}|^2(\xi_x,0,t) + \eta \| PR_2 \|_{2,2}^2 + \| \nu^{\frac12}(I-P)R_2 \|_{2,2}^2 + \frac{1}{\varepsilon^2} \| \nu^{\frac12}(I-P_J)R_2 \|_{2,2}^2 + \varepsilon^2 \| f^- \|_{2\sim}^2 \Big). \tag{4.27}$$

We now discuss the moment $\hat{R}_{20}(\xi_x,0)$ for $\xi_x \neq 0$. For $\varepsilon|\sigma| > \sigma_1$ and σ_1 sufficiently large, consider the equation (4.7) written in Fourier variables for the time and x-dependence. Introduce also the cutoff function β as in Lemma 4.1. Use the term $i\varepsilon\sigma\mathcal{F}_t\mathcal{F}_x\beta R_2(\sigma,\xi_x,z,v)$ to express the ψ_0-moment. For this, project the equation along $v_z - c_2v_z\bar{A}$, and along $c_3\psi_0 + v_x^2\bar{B}$ with $c_3 = -(v_x^2, v_x^2\bar{B}) > 0$ to remove a $\mathcal{F}_t\mathcal{F}_x\beta R_{2v_x}$-moment. That leads to an expression for

$$\mathcal{F}_t\mathcal{F}_x\beta(-i\varepsilon\sigma(R_{2v_z} - c_2R_{2v_z\bar{A}}) - i\mu\xi_x\zeta_1 + \frac{\partial}{\partial z}(R_{20}+\zeta_2)),$$

and for

$$\mathcal{F}_t\mathcal{F}_x\beta\big(- i\varepsilon\sigma(c_3R_{20} + R_{2v_x^2\bar{B}}) - i\mu\xi_x\zeta_3 + \frac{\partial}{\partial z}\big(\frac32c_3R_{2v_z} + \zeta_4)\big),$$

with $\zeta_j,\ j=1,...,4$ certain nonhydrodynamic moments of R_2. Thus we get an expression for

$$-i\varepsilon\sigma c_3\Big(\frac{3}{2}|\mathcal{F}_t\mathcal{F}_x\beta R_{20}|^2+|\mathcal{F}_t\mathcal{F}_x\beta R_{2v_z}|^2\Big) +\frac{\partial}{\partial z}\Big((\mathcal{F}_t\mathcal{F}_x\beta(R_{20}+\zeta_2))(\frac{3}{2}c_3\mathcal{F}_t\mathcal{F}_x\beta(R_{2v_z}+\zeta_4))^*\Big).$$

After division by σ and integration, the boundary term is multiplied by the small coefficient σ_1^{-1}. This can now be estimated separately at $-\pi$ and at π using Green's formula (4.10). It follows that

$$\| (1-\chi_{\sigma_1})\mathcal{F}_t\mathcal{F}_xR_{20}(\,\cdot\,,\xi_x,\,\cdot\,)\|_{2,2}^2\leq C\Big(\sigma_1^{-1}(\|\mathcal{F}_xR_2(\,\cdot\,,\xi_x,\,\cdot\,)\|_{2,2,2}^2 +\frac{1}{\varepsilon^2}\|(I-P_j)\mathcal{F}_xR_2(\,\cdot\,,\xi_x,\,\cdot\,)\|_{2,2,2}^2)+\|(I-P)\mathcal{F}_xR_2(\,\cdot\,,\xi_x,\,\cdot\,)\|_{2,2,2}^2\Big). \tag{4.28}$$

The case of $\varepsilon\sigma$ small requires a different argument. Consider equation (4.7) and its Fourier transform in t, x and z. We denote the total Fourier transform of a function h, $\mathcal{F}_t\mathcal{F}_x\mathcal{F}_zh$ by $\hat{h}$, and by $\hat{h}^z$ the Fourier transform $\mathcal{F}_t\mathcal{F}_xh$. With β defined in Lemma 4.1 put $\overline{\hat{R}_2}:=\widehat{\beta R_2}$ and $\overline{\hat{R}_2}^z:=\widehat{\beta R_2}^z$. We have

$$\begin{aligned}
&\varepsilon i\sigma\overline{\hat{R}_2}+\mu v_xi\xi_x\overline{\hat{R}_2}+iv_z\xi_z\overline{\hat{R}_2}+v_z\overline{\hat{r}}(-1)^{\xi_z}=\\
&\varepsilon M^{-1}G\partial_{v_z}(M\overline{\hat{R}_2})+\varepsilon^{-1}\widehat{L_J\beta R_2}+\beta\widehat{H_1(R_2)}+\varepsilon\widehat{R_2\partial_t\beta}:=\mathcal{N}.\\
&\varepsilon i\sigma\overline{\hat{R}_2}^z+\mu v_xi\xi_x\overline{\hat{R}_2}^z+v_z\partial_z\overline{\hat{R}_2}^z=\\
&\varepsilon M^{-1}G\partial_{v_z}(M\overline{\hat{R}_2})+\varepsilon^{-1}\widehat{L_J\beta R_2}^z+\beta\widehat{H_1(R_2)}^z+\varepsilon\widehat{R_2\partial_t\beta}^z:=\mathcal{N}^z.
\end{aligned} \tag{4.29}$$

We notice that the right-hand sides contain only terms that can be estimated by contributions either involving the nonhydrodynamic part or the hydrodynamic one multiplied by a small factor.

For $\xi_z=0$ we have

$$\varepsilon i\sigma\overline{\hat{R}_2}+\mu v_xi\xi_x\overline{\hat{R}_2}+v_z\overline{\hat{r}}=\mathcal{N}(\sigma,\xi_x,0,v). \tag{4.30}$$

We take the integral $\int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq$ of (4.29)

$$-\int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq\big[\varepsilon\sigma\overline{\hat{R}_2}^z+\mu\xi_xv_x\overline{\hat{R}_2}^z\big]+iv_z\overline{\hat{R}_2}(\sigma,\xi_x,0)-2\pi iv_zr(-\pi) =\int_{-\pi}^{\pi}dz\int_{-\pi}^{z}dq\mathcal{N}^z. \tag{4.31}$$

Let $W(|v|)$ be a smooth function such that

$$\begin{aligned}
&\int_0^\infty\rho^3(W'(\rho))^2M(\rho)d\rho<\infty,\quad \int_0^\infty\rho^5W(\rho)M(\rho)d\rho=1,\\
&\int_0^\infty\rho^6W(\rho)M(\rho)d\rho=0,\\
&\int_0^\infty\rho^7W(\rho)M(\rho)d\rho=3,\quad \int_0^\infty\rho^8W(\rho)M(\rho)d\rho=0.
\end{aligned} \tag{4.32}$$

Then, multiply (4.31) by $v_x v_z M W(|v|)$ and integrate over v. The first term does not contribute to the hydrodynamic part of $\overline{\hat{R}_2^z}$. The contribution to the hydrodynamic part from the second and third terms are respectively

$$\int_{-\pi}^{\pi} dz \int_{-\pi}^{z} dq(\psi_3, \overline{\hat{R}_2}^z)\mu\xi_x \int dv v_x^2 v_z^2 M(|v|)W(|v|),$$

$$2\pi i(\psi_1, \overline{\hat{R}_2}) \int dv v_x^2 v_z^2 M(|v|)W(|v|).$$

Let us use the polar coordinates to compute the v-integrals

$$\int dv v_x^2 v_z^2 M(|v|)W(|v|) = \int_{S_2} d\omega \omega_x^2 \omega_z^2 \int_0^\infty d\rho \rho^6 M(\rho)W(\rho) = 0.$$

The vanishing of the integral is due to the second condition in (4.32). And so on the left-hand side there are only boundary and nonhydrodynamic terms. We put the latter on the right-hand side, denoted now by $\mathcal{N}_1^z$,

$$\int_{v_z<0} dv v_x v_z^2 M(|v|)W(|v|)\gamma^- \overline{\hat{R}_2}(-\pi) = \mathcal{N}_1^z.$$

The v-integral of the ingoing part of r times $v_x v_z^2 M(|v|)W(|v|)$ is zero because of the boundary conditions. In this way we have reached a control of the boundary term $\gamma^- \overline{\hat{R}_2}(-\pi)$. Now we reproduce this term by multiplying (4.30) by $v_x v_z MW$ and integrating over $v_z < 0$,

$$i\varepsilon\sigma \overline{\hat{\bar{R}}}_{2v_x}(\sigma, \xi_x, 0)c_1 + i\mu\xi_x \hat{\bar{R}}_{20}(\sigma, \xi_x, 0)c_1$$

$$+ \int_{v_z<0} dv v_x v_z^2 M(|v|)W(|v|)\gamma^- \hat{\bar{R}}_2(-\pi) = \mathcal{N}_1,$$

where $\mathcal{N}_1$ incorporates $\mathcal{N}$ and all the other nonhydrodynamic terms and

$$c_1 = \int_{S_2^-} d\omega \omega_x^2 \omega_z \int_0^\infty d\rho \rho^5 M(\rho)W(\rho) = \int_{S_2^-} d\omega \omega_x^2 \omega_z,$$

where $S_2^- = \{\omega \in S_2, \omega_z < 0\}$, because of the first condition in (4.32). We have used the second condition in (4.32) to cancel the remaining hydrodynamic terms and as before the ingoing part does not contribute. The conclusion is

$$c_1 \left[\varepsilon\sigma \overline{\hat{\bar{R}}}_{2v_x}(\sigma, \xi_x, 0) + \mu\xi_x \hat{\bar{R}}_{20}(\sigma, \xi_x, 0)\right] = \mathcal{N}_1 - \mathcal{N}_1^z. \tag{4.33}$$

We will get now a second equation involving $\overline{\hat{R}}_{2v_x}$ and $\overline{\hat{R}}_{20}$ with a similar procedure. The two equations together will give us the wanted estimate for these terms.

Multiply (4.31) by $v_x^2 v_z W(|v|)$ and integrate over v. The first and second term do not contribute to the hydrodynamic part of $\hat{R}_2$. The contribution to the hydrodynamic part of the third term is

$$i(\psi_4 + \psi_0, \hat{R}_2) \int dv v_x^2 v_z^2 (\psi_4 + \psi_0) M(|v|)W(|v|).$$

The v-integrals in polar coordinates vanish because of the second and fourth conditions in (4.32). The boundary integral is then given as before in terms of a r.h.s., denoted by $\mathcal{N}_2^z$, involving non-hydrodynamic terms and $\mathcal{N}^z$,

$$i\int_{v_z<0} dv v_x^2 v_z^2 M(|v|)W(|v|)\bar{\hat{R}}_2(-\pi) = \mathcal{N}_2^z. \tag{4.34}$$

The ingoing part vanishes this time because of the second condition in (4.32). Now multiply (4.30) by $v_x^2 v_z MW$ and integrate over $v_z < 0$,

$$\mu\xi_x\overline{\hat{R}}_{2v_x}(\sigma,\xi_x,0)c_2 + \varepsilon\sigma\overline{\hat{R}}_{20}(\sigma,\xi_x,0)c_1$$
$$+\int_{v_z<0} v_x^2 v_z^2 M(|v|)W(|v|)\bar{\hat{R}}_2(-\pi)dv = \mathcal{N}_2,$$

where

$$c_2 = \int_{S_2^-} d\omega\omega_x^4\omega_z \int_0^\infty d\rho\rho^7 M(\rho)W(\rho) = 3\int_{S_2^-} d\omega\omega_x^4\omega_z$$

because of the first and third conditions in (4.32). This time the ingoing part

$$i\int_{v_z<0} dv v_x^2 v_z^2 M(|v|)W(|v|)r_{in}(\pi)$$

gives a contribute of order ε due to the presence of the Maxwellian M_+ in the boundary conditions. We put this term and all the other non-hydrodynamic terms in the r.h.s. term denoted by $\mathcal{N}_2$. By using (4.34) we finally get

$$c_2\varepsilon\sigma\overline{\hat{R}}_{2v_x}(\sigma,\xi_x,0) + c_1\mu\xi_x\overline{\hat{R}}_{20}(\sigma,\xi_x,0) = \mathcal{N}_2 - \mathcal{N}_2^z. \tag{4.35}$$

Equations (4.33) and (4.35) give a system of two equations that allows to express $\hat{R}_{2v_x}$ and $\hat{R}_{20}$ in terms of quantities under control provided that $c_1 \neq c_2$. This can be easily checked by direct computation.

As an end result, it holds for $\varepsilon\sigma \leq \sigma_1$ that

$$\| \chi_{\sigma_1}\hat{R}_{20}(\cdot,\xi_x,0) \|_2^2 \leq c\Big(\frac{1}{\varepsilon^2} \| (I-P_J)R_2 \|_{2,2,2}^2 + \| (I-P)R_2 \|_{2,2,2}^2 + \eta \| R_2 \|_{2,2,2}^2 \Big). \tag{4.36}$$

The same estimate also holds for the ψ_0-moment when $\xi_z \neq 0$. The proof is simpler. The boundary term is this time removed by subtracting the same equation for $\xi_z = 0$ (times $(-1)^{\xi_z}$) and then multiplying first by $v_x v_z MW$ and then by $v_x^2 v_z MW$. This gives two equations

$$\varepsilon\sigma c_1[\bar{\hat{R}}_{2v_x}]_- + c_1\mu\xi_x[\bar{\hat{R}}_{20}]_- = B, \quad \varepsilon\sigma c_2[\bar{\hat{R}}_{2v_x}]_- + c_1\mu\xi_x[\bar{\hat{R}}_{20}]_- = B_1,$$

where $[f]_- = f(\sigma,\xi_x,\xi_z,v) - (-1)^{\xi_z} f(\sigma,\xi_x,0,v)$.

The lemma follows by collecting the previous estimates. □

The above study of R_2 leads to

Lemma 4.6 *Any solution R_2 to the problem (4.7) satisfies the a priori estimates*

$$\begin{aligned}
\| \nu^{\frac{1}{2}}(I-P_J)R_2 \|^2_{2,2,2} &\leq c\Big(\varepsilon \| R_0 \|^2_{2,2} + \varepsilon \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2,2} \\
&+ \frac{1}{\varepsilon} \| P_J g \|^2_{2,2,2} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{2,2,\sim} \Big), \\
\| P_J R_2 \|^2_{2,2,2} &\leq c\Big(\frac{1}{\varepsilon}(\| R_0 \|^2_{2,2} + \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2,2}) \\
&+ \frac{1}{\varepsilon^3} \| P_J g \|^2_{2,2,2} + \frac{1}{\varepsilon^4} \| \bar{\psi} \|^2_{2,2,\sim} \Big), \\
\| \nu^{\frac{1}{2}} R_2 \|^2_{\infty,\infty,2} &\leq c(\frac{1}{\varepsilon^2} \| R_2 \|^2_{\infty,2,2} + \| \gamma^- R_1 \|^2_{\infty,2,\sim} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{\infty,2,\sim}) \\
&\leq c\Big(\frac{1}{\varepsilon^3} \| R_0 \|^2_{2,2} + \frac{1}{\varepsilon^3} \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2,2} \\
&+ \frac{1}{\varepsilon^5} \| P_J g \|^2_{2,2,2} + \frac{1}{\varepsilon^6} \| \bar{\psi} \|^2_{2,2,\sim} + \| R_0 \|^2_{\infty,2} \\
&+ \varepsilon^2 \| \nu^{-\frac{1}{2}} g \|^2_{\infty,\infty,2} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{\infty,2,\sim} \Big).
\end{aligned}$$

Theorem 4.1 *There exists a solution R to the rest term problem (4.1) such that*

$$\int_0^{+\infty} \int_{[-\pi,\pi]} \int_{[-\pi,\pi]} \int_{\mathbb{R}^3} |R(t,x,z,v)|^2 M(v) dt dx dz dv < c\varepsilon^7. \tag{4.37}$$

<u>Proof of Theorem 4.1</u> Take the asymptotic expansion of fifth order in ε. We shall prove that R can be obtained as the limit of an approximating sequence, and that R satisfies (4.37). Since R is a solution to the initial boundary value problem for the rescaled rest term, (4.37) in turn implies the L^2_M-convergence to zero of $R(t)$, when time tends to infinity.
Let the approximating sequence $\{R^n\}$ be defined by $R^0 = 0$, and

$$\begin{aligned}
&\frac{\partial R^{n+1}}{\partial t} + \frac{1}{\varepsilon} v^\mu \cdot \nabla R^{n+1} - GM^{-1} \frac{\partial (MF^{n+1})}{\partial v_z} = \frac{1}{\varepsilon^2} L_J R^{n+1} + \frac{1}{\varepsilon} H_1(R^{n+1}) \\
&\qquad + \frac{1}{\varepsilon} J(R^n, R^n) + A, \\
&R^{n+1}(0,x,z,v) = R_0(x,z,v), \\
&R^{n+1}(t,x,\mp\pi,v) = \frac{M_\mp}{M} \int_{w_z \lesseqgtr 0} (R^{n+1}(t,x,\mp\pi,w) + \frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,w)) |w_z| M dw \\
&\qquad - \frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,v), \quad x \in [-\pi,\pi], \quad t > 0, \ v_z \gtrless 0.
\end{aligned}$$

Here the initial value R_0 is of ε-order four, A has been chosen so that $\int\int\int A(\cdot,x,z,v) M dx dz dv \equiv 0$, $(I-P_J)g = \varepsilon(I-P_J)A$ is of order four,

and $P_J g = \varepsilon P_J A$ is of order five.
The function R^1 is solution to

$$\frac{\partial R^1}{\partial t} + \frac{1}{\varepsilon} v^\mu \cdot \nabla R^1 - GM^{-1} \frac{\partial (MR^1)}{\partial v_z} = \frac{1}{\varepsilon^2} L_J R^1 + \frac{1}{\varepsilon} H_1(R^1) + A,$$

$$R^1(0,x,z,v) = R_0(x,z,v),$$

$$R^1(t,x,\mp\pi,v) = \frac{M}{M_\mp}(v) \int_{w_z \lesseqgtr 0} (R^1(t,x,\mp\pi,w) + \frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,w)) w_z M dw$$
$$-\frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,v), \quad x \in [-\pi,\pi], \quad t > 0, \quad v_z \gtrless 0.$$

Split R^1 into two parts R_1 and R_2, solutions of (4.4) and (4.7), respectively, with $g = \varepsilon A$. Then using the corresponding a priori estimates, Lemma 4.3 and Lemma 4.6 together with the exponential decrease of $\bar{\psi}$, and the ε-orders of R_0 and A, we get for some constant c_1.

$$\| \nu^{\frac{1}{2}} R^1 \|_{\infty,\infty,2} \leq c_1 \varepsilon^{\frac{5}{2}}, \quad \| \nu^{\frac{1}{2}} R^1 \|_{2,2,2} \leq c_1 \varepsilon^{\frac{7}{2}},$$

By induction

$$\| \nu^{\frac{1}{2}} R^j \|_{\infty,\infty,2} \leq 2c_1 \varepsilon^{\frac{5}{2}}, \quad j \leq n+1,$$
$$\| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2,2} \leq c_2 \varepsilon^2 \| \nu^{\frac{1}{2}} (R^n - R^{n-1}) \|_{2,2,2}, \quad n \geq 0,$$

for some constant c_2. Namely, if this holds up to n^{th} order, then

$$\frac{\partial}{\partial t}(R^{n+2} - R^{n+1}) + \frac{1}{\varepsilon} v^\mu \cdot \nabla (R^{n+2} - R^{n+1}) - \frac{G}{M} \frac{\partial}{\partial v_z}(M(R^{n+2} - R^{n+1}))$$
$$= \frac{1}{\varepsilon^2} L_J (R^{n+2} - R^{n+1}) + \frac{1}{\varepsilon} H_1 (R^{n+2} - R^{n+1}) + \frac{1}{\varepsilon} G^{n+1},$$
$$(R^{n+2} - R^{n+1})(0,x,z,v) = 0,$$
$$(R^{n+2} - R^{n+1})(t,x,\mp\pi,v) = \frac{M}{M_\mp} \int_{w_z \lesseqgtr 0} (R^{n+2} - R^{n+1})(t,x,\mp\pi,w) |w_z| M dw$$
$$x \in [-\pi,\pi], \quad t > 0, \ v_z \gtrless 0.$$

Here

$$G^{n+1} = (I-P)G^{n+1} = \tilde{J}(R^{n+1} + R^n, R^{n+1} - R^n).$$

It follows that

$$\| \nu^{\frac{1}{2}} (R^{n+2} - R^{n+1}) \|_{2,2,2} \leq c\varepsilon^{-\frac{1}{2}} \| \nu^{-\frac{1}{2}} G^{n+1} \|_{2,2,2}$$
$$\leq c\varepsilon^{-\frac{1}{2}} \Big(\| \nu^{\frac{1}{2}} R^{n+1} \|_{\infty,\infty,2} + \| \nu^{\frac{1}{2}} R^n \|_{\infty,\infty,2} \Big) \| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2,2}$$
$$\leq c_2 \varepsilon^2 \| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2,2} .$$

Consequently,

$$\| \nu^{\frac{1}{2}} R^{n+2} \|_{2,2,2} \leq \| \nu^{\frac{1}{2}} (R^{n+2} - R^{n+1}) \|_{2,2,2} + \ldots + \| \nu^{\frac{1}{2}} (R^2 - R^1) \|_{2,2,2}$$
$$+ \| \nu^{\frac{1}{2}} R^1 \|_{2,2,2} \leq 2c_1 \varepsilon^{\frac{7}{2}},$$

for ε small enough. Similarly $\| R^{n+2} \|_{\infty,\infty,2} \leq 2c_1\varepsilon^{\frac{5}{2}}$. In particular $\{R^n\}$ is a Cauchy sequence in $L^2_M([0,+\infty) \times \Omega \times \mathbb{R}^3)$. The existence of a solution R to (4.1) follows, and the estimate (4.37) holds. This means that there is a sequence of Lebesgue points in time, $\{t_j\}_{j=1}^{\infty}$ with t_j tending to infinity with j, where the $\| \cdot \|_{2,2}$- norm of the solution R tends to zero. But the $\|R(t,\cdot)\|_{2,2}$ for fixed $t \geq t_j$ is uniformly bounded by the norm at t_j plus some tail integrals from t_j to ∞, hence tends to zero when time tends to infinity. □

This completes the study of the R-term and the stability theorem follows.

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References

1. L. Arkeryd, R.Esposito, R.Marra, A. Nouri *Stability of the Laminar Solution of the Boltzmann Equation for the Benard Problem*, Bull. Academia Sinica, **3** 51–97 (2008).
2. L. Arkeryd, A. Nouri, *Asymptotic techniques for kinetic problems of Boltzmann type*, Proceedings of the 3rd edition of the summer school in "Methods and Models of kinetic theory", Riv. Mat. Univ. Parma **7**, 1–74 (2007).
3. ——— , *On a Taylor-Couette type bifurcation for the stationary nonlinear Boltzmann equation* J. Stat. Phys. **124**, 401-443 (2006).
4. ——— , *An existence result for the stationary Boltzmann equation in a cylindrical geometry*, Arkiv för Matematik **434**, 29-50 (2005).
5. ——— , *The stationary nonlinear Boltzmann equation in a Couette setting: multiple, isolated L^q solutions and positivity*, J. Stat. Phys. **118**, 849-881 (2005).
6. C. Cercignani, R.Esposito, R.Marra, *The Milne problem with a force term*, Transport Theory Stat. Phys. **27**, 1-33 (1998). [1]
7. P.G. Drazin, W.H. Reid, Hydrodynamic Instability, Cambridge Univ. Press, Cambridge, (1981).
8. R. Esposito, J. L. Lebowitz, R. Marra, *Hydrodynamic limit of the stationary Boltzmann equation in a slab*, Comm. Math. Phys. **160**, 49-80 (1994). *The Navier-Stokes limit of stationary solutions of the nonlinear Boltzmann equation*, J. Stat. Phys. 78, 389-412 (1995).
9. R. Esposito, R. Marra, J. L. Lebowitz, *Solutions to the Boltzmann Equation in the Boussinesq Regime*, J. Stat. Phys. **90**, 1129-1178 (1998).
10. C. Foias, O.P. Manley, R. Temam, *Attractors for the Bénard problem: existence and physical bounds on their fractal dimension*, Non-Linear Analysis **11** , 939 – 967 (1987).
11. J.M.Ghidaglia, *Etude d'écoulements fluides visqueux incompressibles : comportement pour les grands temps et applications aux attracteurs*, Thesis, Orsay (1984).
12. V. I. Iudovich, *On the origin of convection*, J. Appl. Math. Mech. **30**, 1193–1199 (1966).

[1] In this paper the assumption on the force has to be modified, asking for a suitable power decay of the force at infinity. This condition is certainly fulfilled in the application to the present case as well as in [9] and [1]. The authors in [6] thank Xiongfeng Yang for pointing them this mistake.

13. ——— *Free convection and bifurcation*, J. Appl. Math. Mech., **31** 103-114 (1967).
14. ——— *Stability of convection flows* J. Appl. Math. Mech., **31**, 272–281 (1967).
15. D. D. Joseph, *On the stability of the Boussinesq equation*, Arch. Rat. Mech. Anal. **20**, 59-71 (1965).
16. O.A. Ladyzenskaja, The boundary value problems of mathematical physics, Applied mathematical sciences Springer-Verlag (1985).
17. N. B. Maslova, Nonlinear evolution equations: kinetic approach, World Scientific, 1993.
18. T. Ma, S. Wang, *Dynamic Bifurcation and Stability in the Rayleigh- Bénard Convection*, Comm. Math. Sci. **2** 159–183 (2004).
19. Y. Sone, Kinetic Theory and Fluid Dynamics, Birkhäuser Boston, 2002; Molecular gas dynamics, theory, techniques, and applications, World Scientific, Birkhäuser Boston, 2007.